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PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

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ERRATUM

PHILOSOPHICAL TRANSACTIONS Vol 171 Part III 1880

Page 1057, at line 20 $\operatorname{Sec}\Gamma\times\left(\surd + 2\,\frac{d\phi}{\surd}\right)$ is put for $\operatorname{Sec}\Gamma\times\left(\surd + 2\,\frac{\phi}{\surd}\right)$

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MDCCCLXXXI

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ADJUDICATION of the MEDALS of the ROYAL SOCIETY for the year 1880, by the PRESIDENT and COUNCIL.

The COPLEY MEDAL to Prof. JAMES JOSEPH SYLVESTER, F.R.S., for his long-continued investigations and discoveries in Mathematics.

A ROYAL MEDAL to Prof. JOSEPH LISTER, F.R.S., for his contributions on various Physiological and Biological Subjects published in the Philosophical Transactions and Proceedings of the Royal Society, and elsewhere; and for his labours, practical and theoretical, on questions relating to the Antiseptic System of Treatment in Surgery.

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PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCLXXX.

VOL. 171.

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LIST OF ILLUSTRATIONS.

XXI. *A Memoir on the Single and Double Theta-Functions.*

By A. CAYLEY, *F.R.S., Sadlerian Professor of Pure Mathematics in the University of Cambridge.*

Received November 14,—Read November 28, 1879.

THE Theta-Functions, although arising historically from the Elliptic Functions, may be considered as in order of simplicity preceding these, and connecting themselves directly with the exponential function (e^x or) exp. x; viz., they may be defined each of them as a sum of a series of exponentials, singly infinite in the case of the single functions, doubly infinite in the case of the double functions; and so on. The number of the single functions is $=4$; and the quotients of these, or say three of them each divided by the fourth, are the elliptic functions sn, cn, dn; the number of the double functions is $(4^2=)$ 16; and the quotients of these, or say fifteen of them each divided by the sixteenth, are the hyper-elliptic functions of two arguments depending on the square root of a sextic function: generally the number of the p-tuple theta-functions is $=4^p$; and the quotients of these, or say all but one of them each divided by the remaining function, are the Abelian functions of p arguments depending on the irrational function y defined by the equation $F(x, y)=0$ of a curve of deficiency p. If instead of connecting the ratios of the functions with a plane curve we consider the functions themselves as coordinates of a point in a (4^p-1)dimensional space, then we have the single functions as the four coordinates of a point on a quadri-quadric curve (one-fold locus) in ordinary space; and the double functions as the sixteen coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space, the deficiency of this two-fold locus being of course $=2$.

The investigations contained in the First Part of the present Memoir, although for simplicity of notation exhibited only in regard to the double functions are, in fact, applicable to the general case of the p-tuple functions; but in the main the Memoir relates only to the single and double functions, and the title has been given to it accordingly. The investigations just referred to extend to the single functions; and there is, it seems to me, an advantage in carrying on the two theories simultaneously up to and inclusive of the establishment of what I call the Product-theorem: this is a natural point of separation for the theories of the single and the double functions respectively. The ulterior developments of the two theories are indeed closely

analogous to each other; but on the one hand the course of the single theory would be only with difficulty perceptible in the greater complexity of the double theory; and on the other hand we need the single theory as a guide for the course of the double theory.

I accordingly stop to point out in a general manner the course of the single theory, and, in connexion with it but more briefly, that of the double theory; and I then, in the Second and Third Parts respectively, consider in detail the two theories separately; first, that of the single functions, and then that of the double functions; the paragraphs of the Memoir are numbered consecutively.

The definition adopted for the theta-functions differs somewhat from that which is ordinarily used.

The earlier memoirs on the double theta-functions are the well-known ones:—

ROSENHAIN, "Mémoire sur les fonctions de deux variables et à quatre périodes, qui sont les inverses des intégrales ultra-elliptiques de la première classe." [1846.] Paris: 'Mém. Savans Étrang.' xi. (1851), pp. 361–468.

GOPEL, 'Theoriæ transcendentium Abelianarum primi ordinis adumbratio levis. 'Crelle,' xxxv. (1847), pp. 277–312.

My first paper—CAYLEY, "On the Double θ-Functions in connexion with a 16-nodal Surface," 'Crelle-Borchardt,' lxxxiii. (1877), pp. 210–219—was founded directly upon these, and was immediately followed by Dr. BORCHARDT's paper,

BORCHARDT, "Ueber die Darstellung der *Kummer*sche Flache vierter Ordnung mit sechzehn Knotenpunkte durch die *Göpel*schen Relation zwischen vier Theta-functionen mit zwei Variabeln." Ditto, pp. 220–233.

My other later papers are contained in the same Journal.

FIRST PART.—INTRODUCTORY.

Definition of the theta-functions.

1. The p-tuple functions depend upon $\frac{1}{2}p(p+1)$ parameters which are the coefficients of a quadric function of p ultimately disappearing integers, upon p arguments, and upon $2p$ characters, each $=0$ or 1, which form the characteristic of the 4^p functions; but it will be sufficient to write down the formulæ in the case $p=2$.

As already mentioned, the adopted definition differs somewhat from that which is ordinarily used. I use, as will be seen, a quadric function $\frac{1}{4}(a, h, b \between m, n)^2$ with *even* integer values of m, n, instead $(a, h, b \between m, n)^2$ with even or odd values; and I write the other term $\frac{1}{2}\pi i(mu+nv)$ instead of $mu+nv$; this comes to affecting the arguments u, v with a factor πi, so that the quarter periods (instead of being πi) are made to be $=1$.

2. We write

$$\begin{pmatrix} m, n \\ u, v \end{pmatrix} = \tfrac{1}{4}(a, h, b \between m, n)^2 + \tfrac{1}{2}\pi i(mu+nv),$$

and in like manner

$$\begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix} = \tfrac{1}{4}(a,\ h,\ b \between m+\alpha,\ n+\beta)^2 + \tfrac{1}{2}\pi i \{(m+\alpha)(u+\gamma)(n+\beta)(v+\delta)\},$$

and prefixing to either of these the functional symbol exp. we have the exponential of the function in question, that is, e with the function as an exponent.

We then write, as the definition of the double theta-functions,

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u,\ v) = \Sigma \exp \begin{pmatrix} m+\alpha, & n+\beta \\ u+\gamma, & v+\delta \end{pmatrix},$$

where the summation extends to all positive and negative even integer values (zero included) of m and n respectively . α, β, γ, δ might denote any quantities whatever, but for the theta-functions they are regarded as denoting positive or negative integers; this being so, it will appear that the only effect of altering each or any of them by an even integer is to reverse (it may be) the sign of the function ; and the distinct functions are consequently the $(4^2=)16$ functions obtained by giving to each of the quantities α, β, γ, δ the two values 0 and 1 successively.

3. We thus have the double theta-functions depending on the parameters $(a,\ h,\ b)$ which determine the quadric function $(a,\ h,\ b \between m,\ n)^2$ of the disappearing even integers $(m,\ n)$: and on the two arguments $(u,\ v)$: in the symbol $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, which is called the characteristic, the characters α, β, γ, δ are each of them $=0$ or 1 , and we thus have the 16 functions.

The parameters $(a,\ h,\ b)$ may be real or imaginary, but they must be such that reducing each of them to its real part the resulting function $(* \between m,\ n)^2$ is invariable in its sign, and negative for all real values of m and n : this is in fact the condition for the convergency of the series which give the values of the theta-functions.

4. The characteristic $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ is said to be even or odd according as the sum $\alpha\gamma+\beta\delta$ is even or odd.

Allied functions.

5. As already remarked, the definition of

$$\vartheta\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}(u,\ v)$$

is not restricted to the case where the α, β, γ, δ represent integers, and there is actually occasion to consider functions of this form where they are not integers : in particular, α, β may be either or each of them of the form, integer $+\tfrac{1}{2}$. But the functions thus obtained *are not regarded as theta-functions,* and the expression theta-function will consequently not extend to include them.

Properties of the theta-functions · Various sub-headings.

Even-integer alteration of characters.

6. If x, y be integers, then m, n having the several even integer values from $-\infty$ to $+\infty$ respectively, it is obvious that $m+\alpha+2x$, $n+\beta+2y$ will have the same series of values with $m+\alpha$, $n+\beta$ respectively; and it thence follows that

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma &, \delta\end{pmatrix}(u, v)=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v).$$

Similarly if z, w are integers, then in the function

$$\vartheta\begin{pmatrix}\alpha &, \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v)$$

the argument of the exponential function contains the term

$$\tfrac{1}{2}\pi i\{m+\alpha.u+\gamma+2z.+.n+\beta.v+\delta+2y\};$$

this differs from its original value by

$$\tfrac{1}{2}\pi i(m+\alpha.2z.+.n+\beta.2w),$$
$$=\pi i(mz+nw)+\pi i(\alpha z+\beta w),$$

and then, m and n being even integers, $mz+nw$ is also an even integer, and the term $\pi i(mz+nw)$ does not affect the value of the exponential: we thus introduce into each term of the series the factor exp. $\pi i(\alpha z+\beta w)$, which is in fact $=(-)^{\alpha z+\beta w}$; and we consequently have

$$\vartheta\begin{pmatrix}\alpha &, \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v)=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v);$$

or, uniting the two results,

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v)=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u, v)\cdot$$

this sustains the before-mentioned conclusion that the only distinct functions are the 16 functions obtained by giving to the characters α, β, γ, δ the values 0 and 1 respectively.

Odd-integer alteration of characters.

7. The effect is obviously to interchange the different functions.

Even and odd functions.

8. It is clear that $-m-\alpha$, $-n-\beta$ have precisely the same series of values with $m+\alpha$, $n+\beta$ respectively: hence considering the function

$$\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(-u, -v)$$

the linear term in the argument of the exponential may be taken to be

$$\tfrac{1}{2}\pi i\{-m-\alpha.-u+\gamma.+.-n-\beta.-v+\delta\},$$

which is

$$=\tfrac{1}{2}\pi i\{m+\alpha.u+\gamma.+.n+\beta.v+\delta\}-\pi i\{m+\alpha.\gamma.+.n+\beta.\delta\};$$

the second term is here $=-\pi i(m\gamma+n\delta)-\pi i(\alpha\gamma+\beta\delta)$, where $m\gamma+n\delta$ being an even integer the part $-\pi i(m\gamma+n\delta)$ does not alter the value of the exponential: the effect of the remaining part $-\pi i(\alpha\gamma+\beta\delta)$ is to affect each term of the series with the factor exp. $-\pi i(\alpha\gamma+\beta\delta)$, or what is the same thing, exp. $\pi i(\alpha\gamma+\beta\delta)$, each of these being in fact $=(-)^{\alpha\gamma+\beta\delta}$.

We have thus

$$\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(-u, -v)=(-)^{\alpha\gamma+\beta\delta}\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u, v),$$

viz., $\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u, v)$ is an even or odd function of the two arguments (u, v) conjointly, according as the characteristic $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$ is even or odd.

The quarter-periods unity.

9. Taking z and w integers, we have from the definition

$$\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u+z, v+w)=\vartheta\begin{pmatrix}\alpha &, \beta\\ \gamma+z, & \delta+w\end{pmatrix}(u, v),$$

viz., the effect of altering the arguments u, v into $u+z$, $v+w$ is simply to interchange the functions as shown by this formula.

If z and w are each of them even, then replacing them by $2z$, $2w$ respectively, we have

$$\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u+2z, v+2w)=\vartheta\begin{pmatrix}\alpha &, \beta\\ \gamma+2z, & \delta+2w\end{pmatrix}(u, v),$$

which by a preceding formula is

$$=(-)^{\alpha z+\beta w}\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u, v),$$

or the function is altered at most in its sign. And again writing $2z$, $2w$ for z, w we have

$$\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u+4z,\ v+4w)=\vartheta\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}(u, v).$$

In reference to the foregoing results we say that the theta-functions have the quarter-periods (1, 1), the half-periods (2, 2), and the whole periods (4, 4).

The conjoint quarter quasi-periods.

10. Taking x, y integers, we consider the effect of the change, u, v into

$$u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by).$$

It is convenient to start from the function

$$\vartheta\begin{pmatrix}\alpha-x, \beta-y\\ \gamma\quad, \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right);$$

the argument of the exponential is here

$$\tfrac{1}{4}(a, h, b\between m+\alpha-x,\ n+\beta-y)^2$$
$$+\tfrac{1}{2}\pi i\left\{m+\alpha-x.u+\gamma+\frac{1}{\pi i}(ax+hy).+.n+\beta-y.v+\delta+\frac{1}{\pi i}(hx+by)\right\}$$

which is

$$=\tfrac{1}{4}(a, h, b\between m+\alpha,\ n+\beta)^2+\tfrac{1}{2}\pi i(m+\alpha.u+\gamma.+.n+\beta.v+\delta)$$

+ other terms which are as follows: viz., they are

$$\begin{array}{ll} -\tfrac{1}{2}(a, h, b\between m+\alpha,\ n+\beta\between x, y) & +\tfrac{1}{2}(m+\alpha.ax+hy.+.n+\beta.hx+by) \\ +\tfrac{1}{4}(a, h, b\between x, y)^2 & -\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta) \\ & -\tfrac{1}{2}(x.ax+hy.+.y.hx+by), \end{array}$$

where the terms of the right hand column are in fact

$$\begin{aligned} &=+\tfrac{1}{2}(a, h, b\between m+\alpha,\ n+\beta\between x, y)\\ &\quad-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\\ &\quad-\tfrac{1}{2}(a, h, b\between x, y)^2, \end{aligned}$$

and the other terms in question thus reduce themselves to

$$-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2 \qquad -\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta),$$

which are independent of m, n, and they thus affect each term of the series with the same exponential factor. The result is

$$\vartheta\begin{pmatrix}\alpha-x, & \beta-y\\ \gamma & , \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy),\ v+\frac{1}{\pi i}(hx+by)\right)$$
$$=\exp\{-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\}.\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v);$$

or (what is the same thing) for α, β, writing $\alpha+x$, $\beta+y$ respectively, we have

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma & \delta\end{pmatrix}\left(u+\frac{1}{\pi i}(ax+hy), v+\frac{1}{\pi i}(hx+by)\right)$$
$$=\exp\{-\tfrac{1}{4}(a,\ h,\ b \between x,\ y)^2-\tfrac{1}{2}\pi i(x.u+\gamma.+.y.v+\delta)\}.\vartheta\begin{pmatrix}\alpha+x, & \beta+y\\ \gamma & , \delta\end{pmatrix}(u,\ v).$$

Taking x, y even, or writing $2x$, $2y$ for x, y, then on the right hand side we have

$$\vartheta\begin{pmatrix}\alpha+2x, & \beta+2y\\ \gamma & , \delta\end{pmatrix}(u,\ v),\ \text{which is}=\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u,\ v),$$

but there is still the exponential factor.

11. The formulæ show that the effect of the change u, v into $u+\frac{1}{\pi i}(ax+hy)$, $v+\frac{1}{\pi i}(hx+by)$, where x, y are integers, is to interchange the functions, affecting them however with an exponential factor; and we hence say that $\frac{1}{\pi i}(a,\ h)$, $\frac{1}{\pi i}(h,\ b)$ are conjoint quarter quasi-periods.

The product-theorem.

12. We multiply two theta-functions

$$\vartheta\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}(u+u',\ v+v'),\ \vartheta\begin{pmatrix}\alpha', & \beta'\\ \gamma', & \delta'\end{pmatrix}(u-u',\ v-v');$$

it is found that the result is a sum of four products

$$\Theta\begin{pmatrix}\tfrac{1}{2}(\alpha+\alpha')+p, & \tfrac{1}{2}(\beta+\beta')+q\\ \gamma+\gamma' & , \quad \delta+\delta'\end{pmatrix}(2u,\ 2v)\quad . \quad \Theta\begin{pmatrix}\tfrac{1}{2}(\alpha-\alpha')+p, & \tfrac{1}{2}(\beta-\beta')+q\\ \gamma-\gamma' & , \quad \delta-\delta'\end{pmatrix}(2u',\ 2v'),$$

where p, q have in the four products respectively the values (0, 0), (1, 0), (0, 1), and (1, 1); Θ is written in place of ϑ to denote that the parameters (a, h, b) are to be changed into $(2a, 2h, 2b)$. It is to be noticed that if α, α' are both even or both odd then $\frac{1}{2}(\alpha+\alpha)$, $\frac{1}{2}(\alpha-\alpha')$ are integers; and so if β, β' are both even or both odd then $\frac{1}{2}(\beta+\beta')$, $\frac{1}{2}(\beta-\beta')$ are integers; and these conditions being satisfied (and in particular they are so if $\alpha=\alpha'$, $\beta=\beta'$) then the functions on the right hand side of the equation are theta-functions (with new parameters as already mentioned); but if the conditions are not satisfied, then the functions on the right hand side are only allied functions. In the applications of the theorem the functions on the right hand side are eliminated between the different equations, as will appear.

13. The proof is immediate: in the first of the theta-functions the argument of the exponential is

$$\begin{pmatrix} m+\alpha & , n+\beta \\ u+u'+\gamma, & v+v'+\delta \end{pmatrix},$$

and in the second, writing m', n' instead of m, n, the argument is

$$\begin{pmatrix} m'+\alpha' & , n'+\beta' \\ u-u'+\gamma', & v-v'+\delta' \end{pmatrix},$$

hence in the product, the argument of the exponential is the sum of these two functions,

$$\begin{aligned} &=\tfrac{1}{4}(a, h, b\between m+\alpha,\ n+\beta)^2+\tfrac{1}{2}\pi i(m+\alpha\,.u+u'+\gamma\,.+.n+\beta\,.v+v'+\delta) \\ &+\tfrac{1}{4}(a, h, b\between m'+\alpha', n'+\beta')^2+\tfrac{1}{2}\pi i(m'+\alpha'.u-u'+\gamma'.+.n'+\beta'.v-v'+\delta'). \end{aligned}$$

Comparing herewith the sum of the two functions

$$\begin{pmatrix} \mu+\frac{1}{2}(\alpha+\alpha'), & \nu+\frac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' & , 2v+\delta+\delta' \end{pmatrix}, \quad \begin{pmatrix} \mu'+\frac{1}{2}(\alpha-\alpha'), & \nu'+\frac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' & , 2v'+\delta-\delta' \end{pmatrix},$$

$$\begin{aligned} &=\tfrac{1}{4}(2a, 2h, 2b\between \mu+\tfrac{1}{2}(\alpha+\alpha'), \nu+\tfrac{1}{2}(\beta+\beta'))^2 \\ &\qquad +\tfrac{1}{2}\pi i\{\mu+\tfrac{1}{2}(\alpha+\alpha').2u+\gamma+\gamma'.+.\nu+\tfrac{1}{2}(\beta+\beta').2v+\delta+\delta'\} \\ &+\tfrac{1}{4}(2a, 2h, 2b\between \mu'+\tfrac{1}{2}(\alpha-\alpha'), \nu'+\tfrac{1}{2}(\beta-\beta'))^2 \\ &\qquad +\tfrac{1}{2}\pi i\{\mu'+\tfrac{1}{2}(\alpha-\alpha').2u'+\gamma-\gamma'.+.\nu'+\tfrac{1}{2}(\beta-\beta').2v'+\delta-\delta'\}, \end{aligned}$$

the two sums are identical if only

$$\begin{aligned} m+m'=2\mu,\ n+n'=2\nu, \\ m-m'=2\mu',\ n-n'=2\nu', \end{aligned}$$

as may easily be verified by comparing the quadric and linear terms separately. The product of the two theta-functions is thus

$$=\Sigma \exp \begin{pmatrix} \mu+\tfrac{1}{2}(\alpha+\alpha'), \nu+\tfrac{1}{2}(\beta+\beta') \\ 2u+\gamma+\gamma' \quad, 2v+\delta+\delta' \end{pmatrix} . \ \Sigma \exp \begin{pmatrix} \mu'+\tfrac{1}{2}(\alpha-\alpha'), \nu'+\tfrac{1}{2}(\beta-\beta') \\ 2u'+\gamma-\gamma' \quad, 2v'+\delta-\delta' \end{pmatrix},$$

with the proper conditions as to the values of μ, ν and of μ', ν' in the two sums respectively. As to this, observe that m, m' are even integers; say for a moment that they are similar when they are both $\equiv 0$ or both $\equiv 2$ (mod 4), but dissimilar when they are one of them $\equiv 0$ and the other of them $\equiv 2$ (mod 4); and the like as regards n, n'. Hence if m, m' are similar μ, μ' are both of them even; but if m, m' are dissimilar then μ, μ' are both of them odd. And so if n, n' are similar, ν, ν' are both of them even, but if n, n' are dissimilar then ν, ν' are both odd.

14. There are four cases

m, m' similar, n, n' similar,
m, m' dissimilar, n, n' similar,
m, m' similar, n, n' dissimilar,
m, m' dissimilar, n, n' dissimilar,

and in the first of these μ, ν, μ', ν' are all of them even, and the product is

$$=\Theta\begin{pmatrix} \tfrac{1}{2}(\alpha+\alpha'), \tfrac{1}{2}(\beta+\beta') \\ \gamma+\gamma', \quad \delta+\delta' \end{pmatrix}(2u, 2v) \ . \ \Theta\begin{pmatrix} \tfrac{1}{2}(\alpha-\alpha'), \tfrac{1}{2}(\beta-\beta') \\ \gamma-\gamma', \quad \delta-\delta' \end{pmatrix}(2u', 2v').$$

In the second case, writing $\mu+1$, $\mu'+1$ for μ, μ' the new values of μ, μ' will be both even, and we have the like expression with only the characters $\tfrac{1}{2}(\alpha+\alpha')$, $\tfrac{1}{2}(\alpha-\alpha')$ each increased by 1; so in the third case we obtain the like expression with only the characters $\tfrac{1}{2}(\beta+\beta')$, $\tfrac{1}{2}(\beta-\beta')$ each increased by 1; and in the fourth case the like expression with the four upper characters each increased by 1. The product of the two theta-functions is thus equal to the sum of the four products, according to the theorem.

Résumé of the ulterior theory of the single functions.

15. For the single theta-functions the Product-theorem comprises 16 equations, and for the double theta-functions, 256 equations: these systems will be given in full in the sequel. But attending at present to the single functions, I write down here the first four of the 16 equations, viz.: these are

$$
\begin{array}{lllll}
0.0 & \vartheta\binom{0}{0}(u+u').\vartheta\binom{0}{0}(u-u') & = & XX'+YY', \\
1.0 & \vartheta\begin{matrix}1\\0\end{matrix} \quad ,, \quad \vartheta\begin{matrix}1\\0\end{matrix} \quad ,, & = & YX'+XY', \\
0.1 & \vartheta\begin{matrix}0\\1\end{matrix} \quad ,, \quad \vartheta\begin{matrix}0\\1\end{matrix} \quad ,, & = & XX'-YY', \\
1.1 & \vartheta\begin{matrix}1\\1\end{matrix} \quad ,, \quad \vartheta\begin{matrix}1\\1\end{matrix} \quad ,, & = & -YX'+XY';
\end{array}
$$

where X, Y denote $\Theta\binom{0}{0}(2u)$, $\Theta\binom{1}{0}(2u)$ respectively, and X′, Y′ the same functions of $2u'$ respectively. In the other equations we have on the left hand the product of *different* theta-functions of $u+u'$, $u-u'$ respectively, and on the right hand expressions involving other functions, X_1, Y_1, X_1', Y_1', &c., of $2u$ and $2u'$ respectively.

16. By writing $u'=0$, we have on the left hand, squares or products of theta-functions of u, and on the right hand expressions containing functions of $2u$: in particular the above equations show that the squares of the four theta-functions are equal to linear functions of X, Y; that is, there exist between the squared functions two linear relations: or again, introducing a variable argument x, the four squared functions may be taken to be proportional to linear functions

$$\mathfrak{A}(a-x),\ \mathfrak{B}(b-x),\ \mathfrak{C}(c-x),\ \mathfrak{D}(d-x)$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, a, b, c, d, are constants. This suggests a new notation for the four functions, viz.: we write

$$
\begin{array}{lllll}
 & \vartheta\binom{0}{0}(u), & \vartheta\binom{1}{0}(u), & \vartheta\binom{0}{1}(u), & \vartheta\binom{1}{1}(u) \\
= & \mathrm{A}u, & \mathrm{B}u, & \mathrm{C}u, & \mathrm{D}u;
\end{array}
$$

and the result just mentioned then is

$$
\begin{array}{c}
\mathrm{A}^2u \quad : \quad \mathrm{B}^2u \quad : \quad \mathrm{C}^2u \quad : \quad \mathrm{D}^2u \\
=\mathfrak{A}(a-x) : \mathfrak{B}(b-x) : \mathfrak{C}(c-x) : \mathfrak{D}(d-x),
\end{array}
$$

which expresses that the four functions are the coordinates of a point on a quadri-quadric curve in ordinary space.

17. The remaining 12 of the 16 equations then contain on the left hand products such as $\mathrm{A}(u+u').\mathrm{B}(u-u')$; and by suitably combining them we obtain equations such as

$$\frac{\overset{u+u'}{\mathrm{B}}\,.\,\overset{u-u'}{\mathrm{A}} - \overset{u+u'}{\mathrm{A}}\,.\,\overset{u-u'}{\mathrm{B}}}{\mathrm{C}\,.\,\mathrm{D} + \mathrm{D}\,.\,\mathrm{C}} = \text{function } (u'),$$

where for brevity the arguments are written above; viz., the numerator of the fraction is

$$\mathrm{B}(u+u')\mathrm{A}(u-u')-\mathrm{A}(u+u')\mathrm{B}(u-u'),$$

and its denominator is

$$\mathrm{C}(u+u')\mathrm{D}(u-u')+\mathrm{D}(u+u')\mathrm{C}(u-u').$$

Admitting the form of the equation, the value of the function of u' is at once found by writing in the equation $u=0$; it is, as it ought to be, a function vanishing for $u'=0$.

18. Take in this equation u' indefinitely small; each side divides by u', and the resulting equation is

$$\frac{\mathrm{A}u\mathrm{B}'u-\mathrm{B}u\mathrm{A}'u}{\mathrm{C}u\mathrm{D}u}=\text{const.}$$

where $\mathrm{A}'u$, $\mathrm{B}'u$ are the derived functions, or differential coefficients in regard to u. It thus appears that the combination $\mathrm{A}u\mathrm{B}'u-\mathrm{B}u\mathrm{A}'u$ is a constant multiple of $\mathrm{C}u\mathrm{D}u$: or, what is the same thing, that the differential coefficient of the quotient-function $\frac{\mathrm{B}u}{\mathrm{A}u}$ is a constant multiple of the product of the two quotient-functions $\frac{\mathrm{C}u}{\mathrm{A}u}$ and $\frac{\mathrm{D}u}{\mathrm{A}u}$.

19. And then substituting for the several quotient-functions their values in terms of x, we obtain a differential relation between x, u; viz.: the form hereof is

$$du=\frac{\mathrm{M}dx}{\sqrt{a-x.b-x.c-x.d-x}},$$

and it thus appears that the quotient-functions are in fact elliptic-functions: the actual values as obtained in the sequel are

$$\begin{aligned}\mathrm{sn}\,\mathrm{K}u&= -\frac{1}{\sqrt{k}}\mathrm{D}u\div\mathrm{C}u,\\ \mathrm{cn}\,\mathrm{K}u&= \sqrt{\frac{k'}{k}}\mathrm{B}u\div\mathrm{C}u,\\ \mathrm{dn}\,\mathrm{K}u&= \sqrt{k'}\mathrm{A}u\div\mathrm{C}u\,;\end{aligned}$$

and we thus of course identify the functions $\mathrm{A}u$, $\mathrm{B}u$, $\mathrm{C}u$, $\mathrm{D}u$ with the H and Θ of JACOBI.

20. If in the above-mentioned four equations we write first $u=0$, and then $u'=0$, and by means of the results eliminate from the original equations the quantities

X, Y, X′, Y′ which occur therein, we obtain expressions for the four products such as $\mathrm{A}(u+u')\mathrm{A}(u-u')$. One of these equations is

$$\mathrm{C}^2 0.\mathrm{C}(u+u')\mathrm{C}(u-u')=\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u'.$$

Taking herein u' indefinitely small, we obtain

$$\frac{\mathrm{C}u\mathrm{C}''u-(\mathrm{C}'u)^2}{\mathrm{C}^2u}=\frac{\mathrm{C}''0}{\mathrm{C}0}-\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2\frac{\mathrm{D}^2u}{\mathrm{C}^2u},$$

where the left hand side is in fact $\frac{d^2}{du^2}\log\mathrm{C}u$, or this second derived function of the theta-function $\mathrm{C}u$ is given in terms of the quotient-function $\frac{\mathrm{D}u}{\mathrm{C}u}$: hence integrating twice, and taking the exponential of each side we obtain $\mathrm{C}u$ as an exponential the argument of which contains the double integral $\iint\frac{\mathrm{D}^2u}{\mathrm{C}^2u}(du)^2$, of a squared quotient-function. This in fact corresponds to JACOBI's equation

$$\Theta u=\sqrt{\frac{2\mathrm{K}k'}{\pi}}e^{\frac{1}{2}u^2\left(1-\frac{\mathrm{E}}{\mathrm{K}}\right)-k^2\int_0 du\int_0 du\,\mathrm{sn}^2u}.$$

21. From the same equation $\mathrm{C}^2 0.\mathrm{C}(u+u')\mathrm{C}(u-u')=\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u'$, differentiating logarithmically in regard to u', and integrating in regard to u, we obtain an equation containing on the left hand side a term $\log\frac{\mathrm{C}(u-u')}{\mathrm{C}(u+u')}$, and on the right hand an integral in regard to u, and which in fact corresponds to JACOBI's equation

$$u\frac{\Theta'a}{\Theta a}+\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}=\Pi(u,a),$$

$$=\int_0\frac{k^2\,\mathrm{sn}\,a\,\mathrm{cn}\,a\,\mathrm{dn}\,a\,\mathrm{sn}^2u\,du}{1-k^2\,\mathrm{sn}^2a\,\mathrm{sn}^2u}.$$

22. It may further be noticed that if in the equation in question, and in the three other equations of the system, we introduce into the integral the variable x in place of u, and the corresponding quantity ξ in place of u', then the integral is that of an expression such as

$$\frac{dx}{\mathrm{T}\sqrt{a-x\,b-x\,c-x\,d-x}},$$

where T is $=x-\xi$, or is $=$ any one of three forms such as

$$\begin{vmatrix}1, & x+\xi, & x\xi\\ 1, & a+b, & ab\\ 1, & c+d, & cd\end{vmatrix}.$$

Résumé of the ulterior theory of the double functions.

23. The ulterior theory of the double functions is intended to be carried out on the like plan. As regards these, it is to be observed here that we have not only the 16 equations leading to linear relations between the squared functions, but that the remaining 240 equations lead also to linear relations between binary products of different functions. We have thus between the 16 functions a system of quadric relations, which in fact determine the ratios of the 16 functions in terms of two variable parameters x, y. (The 16 functions are thus the coordinates of a point on a quadri-quadric two-fold locus in 15-dimensional space.) The forms depend upon six constants, a, b, c, d, e, f: and writing for shortness

$$\sqrt{a} = \qquad \sqrt{a-x.a-y},$$

$$\vdots$$

$$\sqrt{ab} = \frac{1}{x-y}\{\sqrt{a-x.b-x\,f-x.c-y.d-y.e-y} + \sqrt{a-y.b-y\,f-y\,c-x.d-x.e-x}\},$$

$$\vdots$$

(observe that in the symbols $\sqrt{ab}$ it is always f that accompanies the two expressed letters a, b—or, what is the same thing, the duad ab is really an abbreviation for the double triad $abf.cde$); then the 16 functions are proportional to properly determined constant multiples of

$$\sqrt{a},\ \sqrt{b},\ \sqrt{c},\ \sqrt{d},\ \sqrt{e},\ \sqrt{f},\ \sqrt{ab},\ \sqrt{ac},\ \sqrt{ad},\ \sqrt{ae},\ \sqrt{bc},\ \sqrt{bd},\ \sqrt{be},\ \sqrt{cd},\ \sqrt{ce},\ \sqrt{de},$$

and this suggests that the functions shall be represented by the single and double letter notation $A(u. v)$, .. $AB(u, v)$ ... viz., if for shortness the arguments are omitted, then we have

$$A,\ B,\ C,\ D,\ E,\ F,\ AB,\ AC,\ AD,\ AE,\ BC,\ BD,\ BE,\ CD,\ CE,\ DE$$

proportional to determinate constant multiples of the before-mentioned functions $\sqrt{a}$, .. $\sqrt{ab}$, ... of x and y.

24. It is interesting to notice why in the expressions for $\sqrt{ab}$, &c., the sign connecting the two radicals is $+$; the effect of the interchange of x, y is in fact to change (u, v) into $(-u, -v)$; consequently to change the sign of the odd functions, and to leave unaltered those of the even functions: the interchange does in fact leave $\sqrt{a}$, &c., unaltered, while it changes $\sqrt{ab}$, &c., into $-\sqrt{ab}$, &c.; and thus, since only the ratios are attended to, there is a change of sign as there should be.

25. The equations of the product-theorem lead to expressions for

$$\overset{u+u'}{A}\,.\,\overset{u-v'}{B} - \overset{u+u'}{B}\,.\,\overset{u-u'}{A},$$

(where the arguments, written above, are used to denote the *two* arguments, viz.: $u+u'$ to denote $(u+u',\ v+v')$ and $u-u'$ to denote $(u-u',\ v-v')$; and where the letters A, B denote each or either of them a single or double letter) in terms of the functions of (u, v) and of (u', v'): and in any such expression taking u', v' each of them indefinitely small, but with their ratio arbitrary, we obtain the value of

$$\overset{u}{\mathrm{A}}.\delta\overset{u}{\mathrm{B}}-\overset{u}{\mathrm{B}}.\delta\overset{u}{\mathrm{A}},$$

(viz, u here stands for the two arguments (u, v), and δ denotes total differentiation $\delta\mathrm{A}=du\frac{d}{du}\mathrm{A}(u,v)+dv\frac{d}{dv}\mathrm{A}(u,v))$ as a quadric function of the functions of (u, v): or dividing by A^2, the form is $\delta\frac{\mathrm{B}}{\mathrm{A}}=$ a function of the quotient-functions $\frac{\mathrm{B}}{\mathrm{A}}$, &c., that is, we have the differentials of the quotient-functions in terms of the quotient-functions themselves. Substituting for the quotient-functions their values in terms of x, y, we should obtain the differential relations between dx, dy, du, dv, viz., putting for shortness $\mathrm{X}=a-x.b-x.c-x.d-x.e-x.f-x$, and $\mathrm{Y}=a-y.b-y.c-y.d-y.e-y.f-y$, these are of the form

$$\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}},\ \frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}},$$

each of them equal to a linear function of du and dv: so that the quotient-functions are in fact the 15 hyper-elliptic functions belonging to the integrals $\int\frac{dx}{\sqrt{\mathrm{X}}}$, $\int\frac{xdx}{\sqrt{\mathrm{X}}}$; and there is thus an addition-theorem for them, in accordance with the theory of these integrals.

26. The first 16 equations of the product-theorem, putting therein first $u=0$, $v=0$, and then $u'=0$, $v'=0$, and using the results to eliminate the functions on the right hand side, give expressions for

$$\overset{u+u'}{\mathrm{A}}\ .\ \overset{u-u'}{\mathrm{B}},\ \&\text{c.}$$

(that is, $\mathrm{A}(u+u',\ v+v').\mathrm{B}(u-u',\ v-v')$ &c.) in terms of the functions of (u, v) and (u', v'): and we have thus an addition-with-subtraction theorem for the double theta-functions. And we have thence also consequences analogous to those which present themselves in the theory of the single functions.

Remark as to notation.

27. I remark as regards the single theta-functions that the characteristics

$$\binom{0}{0}\quad\binom{1}{0}\quad\binom{0}{1}\quad\binom{1}{1}$$

might for shortness be represented by a series of current numbers

$$0,\qquad 1,\qquad 2,\qquad 3$$

and the functions be accordingly called $\vartheta_0 u$, $\vartheta_1 u$, $\vartheta_2 u$, $\vartheta_3 u$; but that instead of this I prefer to use throughout the before-mentioned functional symbols

$$\text{A}, \qquad \text{B}, \qquad \text{C}, \qquad \text{D}.$$

As regards the double functions, I do, however, denote the characteristics

$$\begin{array}{cccc|cccc|cccc|cccc|} 00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 & 00, & 10, & 01, & 11 \\ 00 & 00 & 00 & 00 & 10 & 10 & 10 & 10 & 01 & 01 & 01 & 01 & 11 & 11 & 11 & 11 \end{array}$$

by a series of current numbers

$$0, \quad 1, \quad 2, \quad 3, \qquad 4, \quad 5, \quad 6, \quad 7, \qquad 8, \quad 9, \quad 10, \quad 11, \qquad 12, \quad 13, \quad 14, \quad 15$$

and write the functions as ϑ, $\vartheta_1 \ldots \vartheta_{15}$ accordingly; and I use also, as and when it is convenient, the foregoing single and double letter notation A, AB, . . , which correspond to them in the order

BD, CE, CD, BE, AC, C, AB, B, BC, DE, F, A, AD, D, E, AE

Moreover I write down for the most part a single argument only: thus, $\text{A}(u+u')$ stands for $\text{A}(u+u', v+v')$, $\text{A}(0)$ for $\text{A}(0, 0)$: and so in other cases.

SECOND PART.—THE SINGLE THETA-FUNCTIONS.

Notation, &c.

28. Writing exp. $a=q$, and converting the exponentials into circular functions, we have directly from the definition

$$\vartheta\tbinom{0}{0}(u)=\vartheta u=\text{A}u=1+2q\cos\pi u+2q^4\cos 2\pi u+2q^9\cos 3\pi u+\ldots\quad,$$

$$\vartheta\tbinom{1}{0}(u)=\vartheta_1 u=\text{B}u=\qquad 2q^{\frac{1}{4}}\cos\tfrac{1}{2}\pi u+2q^{\frac{9}{4}}\cos\tfrac{3}{2}\pi u+2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u+\ldots,$$

$$\vartheta\tbinom{0}{1}(u)=\vartheta_2 u=\text{C}u=1-2q\cos\pi u+2q^4\cos 2\pi u-2q^9\cos 3\pi u+\ldots\ (=\Theta(\text{K}u),\ \text{Jacobi}),$$

$$\vartheta\tbinom{1}{1}(u)=\vartheta_3 u=\text{D}u=\ -2q^{\frac{1}{4}}\sin\tfrac{1}{2}\pi u+2q^{\frac{9}{4}}\sin\tfrac{3}{2}\pi u-2q^{\frac{25}{4}}\cos\tfrac{5}{2}\pi u+\ldots\ (=-\text{H}(\text{K}u),\ \text{Jacobi}),$$

where a is of the form $a=-\alpha+\beta i$, α being non-evanescent and positive: hence $q=\exp.(-\alpha+\beta i)=e^{-\alpha}(\cos\beta+i\sin\beta)$, where $e^{-\alpha}$, the modulus of q is positive and less than 1; $\cos\beta$ may be either positive or negative, and $q^{\frac{1}{4}}$ is written to denote

exp. $\frac{1}{4}(-\alpha+\beta i)$, viz.: this is $=e^{-\frac{1}{4}\alpha}\{\cos\frac{1}{4}\beta+i\sin\frac{1}{4}\beta\}$. But usually $\beta=0$, viz., q is a real positive quantity less than 1, and $q^{\frac{1}{4}}$ denotes the real fourth root of q.

I have given above the three notations, but as already mentioned propose to employ for the four functions the notation Au, Bu, Cu, Du: it will be observed that Du is an odd function, but that Au, Bu, Cu are even functions of u.

The constants of the theory.

29. We have

$$\begin{aligned}
\mathrm{A}0&=1+2q+2q^4+2q^9+\ldots\quad,\\
\mathrm{B}0&=\phantom{1+{}}2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\ldots\quad,\\
\mathrm{C}0&=1-2q+2q^4-2q^9+\ldots\quad,\\
\mathrm{D}0&=0\quad,\\
\mathrm{D}'0&=-\pi\{q^{\frac{1}{4}}-3q^{\frac{9}{4}}+5q^{\frac{25}{4}}-\ldots\}.
\end{aligned}$$

If, as definitions of k, k', K, we assume

$$k=\frac{\mathrm{B}^2 0}{\mathrm{A}^2 0},\ k'=\frac{\mathrm{C}^2 0}{\mathrm{A}^2 0},\ \mathrm{K}=-\frac{\mathrm{A}0}{\mathrm{B}0}\cdot\frac{\mathrm{D}'0}{\mathrm{C}0},$$

then we have

$$\begin{aligned}
k&=4\sqrt{q}\left\{\frac{1+q^2+q^6+\ldots}{1+2q+2q^4+\ldots}\right\}^2,\ =4\sqrt{q}(1-4q+14q^2+\ldots),\\
k'&=\left\{\frac{1-2q+2q^4-\ldots}{1+2q+2q^4+\ldots}\right\}^2,\ =1-8q+32q^2-96q^3+\ldots,\\
\mathrm{K}&=\frac{\pi(1+2q+2q^4+\ldots)(1-3q^2+5q^6-\ldots)}{2(1-2q+2q^4-\ldots)(1+q^2+q^6+\ldots)},\ =\tfrac{1}{2}\pi(1+4q+4q^2+0q^3+\ldots),
\end{aligned}$$

where I have added the first few terms of the expansions of these quantities. We have identically

$$k^2+k'^2=1.$$

It will be convenient to write also as the definition of E,

$$\mathrm{K}(\mathrm{K}-\mathrm{E})=\frac{\mathrm{C}''0}{\mathrm{C}0}:$$

we have then

$$\mathrm{E}=\mathrm{K}-\frac{1}{\mathrm{K}}\frac{\mathrm{C}''0}{\mathrm{C}0},\ =\frac{1}{\mathrm{A}0.\mathrm{B}0.\mathrm{C}0.\mathrm{D}'0}\{-\mathrm{A}^2 0(\mathrm{D}'0)^2+\mathrm{B}^2 0.\mathrm{C}0.\mathrm{C}''0\},$$

and moreover

$$1-\frac{\mathrm{E}}{\mathrm{K}}=\frac{1}{\mathrm{K}^2}\cdot\frac{\mathrm{C}''0}{\mathrm{C}0},\ =\frac{2\pi^2}{\mathrm{K}^2}\cdot\frac{q-4q^4+9q^9-\ldots}{1-2q+2q^4+\ldots},$$

giving

$$\frac{\mathrm{E}}{\mathrm{K}}= \quad 1-8q+48q^2-224q^3+\ldots,$$

and thence

$$\mathrm{E}=\tfrac{1}{2}\pi\{1-4q+20q^2-64q^3+\ldots\}.$$

30. Other formulæ are

$$k=4\sqrt{q}\left\{\frac{1+q^2}{1+q}\frac{1+q^4}{1+q^3}\cdots\right\}^4,$$

$$k'= \quad \left\{\frac{1-q}{1+q}\frac{1-q^3}{1+q^3}\cdots\right\}^4,$$

$$\mathrm{K}= \quad \tfrac{1}{2}\pi\left\{\frac{1+q}{1-q}\frac{1+q^3}{1-q^3}\cdots\frac{1-q^2}{1+q^2}\frac{1-q^4}{1+q^4}\cdots\right\}^2.$$

31. Jacobi's definition of q is from a different point of view altogether, viz., we have $q=\exp.-\frac{\pi\mathrm{K}'}{\mathrm{K}}$, where

$$\mathrm{K}=\int_0^1\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}};$$

and K' is the like function of k', the equation gives $\log q=-\frac{\pi\mathrm{K}'}{\mathrm{K}}$; viz., we have

$$\mathrm{K}'=-\frac{\mathrm{K}}{\pi}\log q,$$

and, regarding herein K as a given function of q, this equation gives K' as a function of q.

The product-theorem.

32. The product-theorem is

$$\vartheta\binom{\alpha}{\gamma}(u+u').\vartheta\binom{\alpha'}{\gamma'}(u-u')=\Theta\binom{\frac{1}{2}(\alpha+\alpha')}{\gamma+\gamma'}2u.\Theta\binom{\frac{1}{2}(\alpha-\alpha')}{\gamma-\gamma'}2u'+\Theta\binom{\frac{1}{2}(\alpha+\alpha')+1}{\gamma+\gamma'}2u.\Theta\binom{\frac{1}{2}(\alpha-\alpha')+1}{\gamma-\gamma'}2u',$$

and here giving to $\begin{smallmatrix}\alpha\\ \gamma\end{smallmatrix}, \begin{smallmatrix}\alpha'\\ \gamma'\end{smallmatrix}$, their different values, and introducing unaccented and accented capitals to denote the functions of $2u$ and $2u'$ respectively, the 16 equations are

A.A	$\vartheta^{0}_{0}\,u+u'\,\vartheta^{0}_{0}\,u-u'=$	$XX'+YY',$	(square-set)
B.B	$\vartheta^{1}_{0}\ ,,\ \vartheta^{1}_{0}\ ,,\ =$	$YX'+XY',$	
C.C	$\vartheta^{0}_{1}\ ,,\ \vartheta^{0}_{1}\ ,,\ =$	$XX'-YY',$	
D.D	$\vartheta^{1}_{1}\ ,,\ \vartheta^{1}_{1}\ ,,\ =$	$-YX'+XY';$	
C.A	$\vartheta^{0}_{1}\,u+u'\,\vartheta^{0}_{0}\,u-u'=$	$X_{,}X'_{,}+Y_{,}Y'_{,},$	(first product-set)
A.C	$\vartheta^{0}_{0}\ ,,\ \vartheta^{0}_{1}\ ,,\ =$	$X_{,}X'_{,}-Y_{,}Y'_{,},$	
D.B	$\vartheta^{1}_{1}\ ,,\ \vartheta^{1}_{0}\ ,,\ =$	$Y_{,}X'_{,}+X_{,}Y'_{,},$	
B.D	$\vartheta^{1}_{0}\ ,,\ \vartheta^{1}_{1}\ ,,\ =$	$Y_{,}X'_{,}-X_{,}Y'_{,};$	
B.A	$\vartheta^{1}_{0}\,u+u'\,\vartheta^{0}_{0}\,u-u'=$	$PP'+QQ',$	(second product-set)
A.B	$\vartheta^{0}_{0}\ ,,\ \vartheta^{1}_{0}\ ,,\ =$	$PQ'+QP',$	
D.C	$\vartheta^{1}_{1}\ ,,\ \vartheta^{0}_{1}\ ,,\ =$	$iPP'-iQQ',$	
C.D	$\vartheta^{0}_{1}\ ,,\ \vartheta^{1}_{1}\ ,,\ =$	$iPQ'-iQP';$	
D.A	$\vartheta^{1}_{1}\,u+u'\,\vartheta^{0}_{0}\,u-u'=$	$P_{,}P'_{,}+Q_{,}Q'_{,},$	(third product-set)
A.D	$\vartheta^{0}_{0}\ ,,\ \vartheta^{1}_{1}\ ,,\ =$	$iP_{,}Q'_{,}-iQ_{,}P'_{,},$	
B.C	$\vartheta^{1}_{0}\ ,,\ \vartheta^{0}_{1}\ ,,\ =$	$-iP_{,}P'_{,}+iQ_{,}Q'_{,},$	
C.B	$\vartheta^{0}_{1}\ ,,\ \vartheta^{1}_{0}\ ,,\ =$	$P_{,}Q'_{,}+Q_{,}P'_{,}.$	

33. Here, and subsequently, we have

$\Theta^{0}_{0},\ \Theta^{1}_{0},\ \Theta^{0}_{1},\ \Theta^{1}_{1}\ (2v)=X,\ Y,\ X_{,},\ Y_{,}$	$\Theta^{\frac{1}{2}}_{0},\ \Theta^{\frac{3}{2}}_{0},\ \Theta^{\frac{1}{2}}_{1},\ \Theta^{\frac{3}{2}}_{1}\ (2u)=P,\ Q,\ P_{,},\ Q_{,}$
$,,\ \ ,,\ \ ,,\ \ ,,\ (2u')=X',\ Y',\ X'_{,},\ Y'_{,}$	$,,\ \ ,,\ \ ,,\ \ ,,\ (2u')=P',\ Q',\ P'_{,},\ Q'_{,}$
$,,\ \ ,,\ \ ,,\ \ ,,\ (0)=\alpha,\ \beta,\ \alpha_{,},\ \beta_{,}$	$,,\ \ ,,\ \ ,,\ \ ,,\ (0)=\mathrm{p},\ \mathrm{q},\ \mathrm{p}_{,},\ \mathrm{q}_{,}$

viz., we use also α, β, $\alpha_{,}$, $\beta_{,}$ and p, q, $\mathrm{p}_{,}$, $\mathrm{q}_{,}$ to denote the zero-functions; $\beta_{,}$ is $=0$, but we use $\beta'_{,}$ to denote the zero-value of $\frac{d}{dv}Y_{,}$.

34. For obtaining the foregoing relations it is necessary to observe that

$$\Theta^{\alpha+2}_{\gamma}=\Theta^{\alpha}_{\gamma};$$

by which the upper character is always reduced to 0, 1, $\frac{1}{2}$ or $\frac{3}{2}$, and that for reducing the lower character we have

$$\Theta^{0}_{\gamma+2}=\Theta^{0}_{\gamma};\ \Theta^{1}_{\gamma+2}=-\Theta^{1}_{\gamma};$$

$$\Theta^{\frac{1}{2}}_{\gamma+2}=i\Theta^{\frac{1}{2}}_{\gamma},\ \Theta^{\frac{1}{2}}_{\gamma-2}=-i\Theta^{\frac{1}{2}}_{\gamma};\ \Theta^{\frac{3}{2}}_{\gamma+2}=-i\Theta^{\frac{3}{2}}_{\gamma},\ \Theta^{\frac{3}{2}}_{\gamma-2}=i\Theta^{\frac{3}{2}}_{\gamma};$$

by means of which the lower character is always reduced to 0 or 1: in all these formulæ the argument is arbitrary, and it is thus $=2u$, or $2u'$ as the case requires. The formulæ are obtained without difficulty directly from the definition of the functions Θ.

35. As an instance, taking $\begin{smallmatrix}\alpha\\ \gamma\end{smallmatrix}, \begin{smallmatrix}\alpha'\\ \gamma'\end{smallmatrix}=\begin{smallmatrix}1\\ 1\end{smallmatrix}, \begin{smallmatrix}0\\ 1\end{smallmatrix}$, the product-equation is

$$\begin{aligned}\vartheta^{1}_{1}(u+u').\vartheta^{0}_{1}(u-u') &= \Theta^{\frac{1}{2}}_{2}(2u).\Theta^{\frac{1}{2}}_{0}(2u')+\Theta^{\frac{3}{2}}_{2}(2u).\Theta^{\frac{3}{2}}_{0}(2u'),\\ &= i\Theta^{\frac{1}{2}}_{0}(2u)\ \Theta^{\frac{1}{2}}_{0}(2u')-i\Theta^{\frac{3}{2}}_{0}(2u).\Theta^{\frac{3}{2}}_{0}(2u'),\\ &= i\mathrm{P}\,\mathrm{P}' \qquad\qquad -i\mathrm{Q}.\mathrm{Q}',\end{aligned}$$

which agrees with the before-given value.

36. The following values are not actually required, but I give them to fix the ideas, and show the meaning of the quantities with which we work.

	$u=0$
$\mathrm{X}=\Theta^{0}_{0}(2u)=1+2q^2\cos 2\pi u+2q^8\cos 4\pi u+\ldots,$	$\alpha=1+2q^2+2q^8+\ldots,$
$\mathrm{Y}=\Theta^{1}_{0}(2u)=\quad 2q^{\frac{1}{2}}\cos\ \pi u+2q^{\frac{9}{2}}\cos 3\pi u+\ldots,$	$\beta=\quad 2q^{\frac{1}{2}}+2q^{\frac{9}{2}}+\ldots,$
$\mathrm{X}_{,}=\Theta^{0}_{1}(2u)=1-2q^2\cos 2\pi u+2q^8\cos 4\pi u+\ldots,$	$\alpha_{,}=1-2q^2+2q^8\ldots,$
$\mathrm{Y}_{,}=\Theta^{1}_{1}(2u)=-2q^{\frac{1}{2}}\sin\ \pi u+2q^{\frac{9}{2}}\sin 3\pi u+\ldots,$	$\beta_{,}'=2\pi(-q^{\frac{1}{2}}+3q^{\frac{9}{2}}-\ldots)$
	$=\frac{d}{du}\mathrm{Y}_{,}$ for $u=0$.

$$P=\Theta_0^{\frac{1}{2}}(2u)=q^{\frac{1}{4}}(\cos\tfrac{1}{2}\pi u+i\sin\tfrac{1}{2}\pi u)+q^{\frac{9}{4}}(\cos\tfrac{3}{2}\pi u-i\sin\tfrac{3}{2}\pi u)$$
$$+q^{\frac{25}{4}}(\cos\tfrac{5}{2}\pi u+i\sin\tfrac{5}{2}\pi u)+\ldots,$$

$$Q=\Theta_0^{\frac{3}{2}}(2u)=q^{\frac{1}{4}}(\cos\tfrac{1}{2}\pi u-i\sin\tfrac{1}{2}\pi u)+q^{\frac{9}{4}}(\cos\tfrac{3}{2}\pi u+i\sin\tfrac{3}{2}\pi u)$$
$$+q^{\frac{25}{4}}(\cos\tfrac{5}{2}\pi u-i\sin\tfrac{5}{2}\pi u)+\ldots,$$

$$P_{\prime}=\Theta_1^{\frac{1}{2}}(2u)=\frac{1+\iota}{\sqrt{2}}\Big\{q^{\frac{1}{4}}(\cos\tfrac{1}{2}\pi u+i\sin\tfrac{1}{2}\pi u)-q^{\frac{9}{4}}(\cos\tfrac{3}{2}\pi u-i\sin\tfrac{3}{2}\pi u)$$
$$-q^{\frac{25}{4}}(\cos\tfrac{5}{2}\pi u+i\sin\tfrac{5}{2}\pi u)++\ldots,$$

$$Q_{\prime}=\Theta_1^{\frac{3}{2}}(2u)=\frac{1-\iota}{\sqrt{2}}\Big\{q^{\frac{1}{4}}(\cos\tfrac{1}{2}\pi i-i\sin\tfrac{1}{2}\pi u)-q^{\frac{9}{4}}(\cos\tfrac{3}{2}\pi u+i\sin\tfrac{3}{2}\pi u)$$
$$-q^{\frac{25}{4}}(\cos\tfrac{5}{2}\pi u-i\sin\tfrac{5}{2}\pi u)++\ldots;$$

and therefore also

$$\mathrm{p}=\mathrm{q}=q^{\frac{1}{4}}+q^{\frac{9}{4}}+q^{\frac{25}{4}}\ldots\ .,$$

$$\mathrm{p}_{\prime}=\frac{1+\iota}{\sqrt{2}}\Big\{q^{\frac{1}{4}}-q^{\frac{9}{4}}-q^{\frac{25}{4}}+q^{\frac{49}{4}}+q^{\frac{81}{4}}--\ldots\Big\},\ \mathrm{q}_{\prime}=\frac{1-\iota}{\sqrt{2}}\Big\{\mathrm{Do.}\Big\};\ \mathrm{p}_{\prime}=i\mathrm{q}_{\prime}$$

The square set, $u'=0$; and x-formulæ.

37. We use the square-set, in the first instance by writing therein $u'=0$; the equations become

$$\begin{aligned}
A^2u&=\alpha X+\beta Y, &&=\omega^2\mathfrak{A}(a-x),\\
B^2u&=\beta X+\alpha Y, &&=\omega^2\mathfrak{B}(b-x),\\
C^2u&=\alpha X-\beta Y, &&=\omega^2\mathfrak{C}(c-x),\\
D^2u&=\beta X-\alpha Y, &&=\omega^2\mathfrak{D}(d-x),
\end{aligned}$$

viz., the equations without their last members show that there exist functions ω^2 and $x\omega^2$, linear functions of X and Y, such that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{A}a$, $\mathfrak{B}b$, $\mathfrak{C}c$, $\mathfrak{D}d$, being constants, the squared functions may be assumed equal to $\mathfrak{A}a.\omega^2-\mathfrak{A}.\omega^2x$, &c., that is, $\omega^2\mathfrak{A}(a-x)$, &c., respectively : the squared functions are then *proportional* to the values $\mathfrak{A}(a-x)$, &c.

To show the meaning of the factor ω^2, observe that from any two of the equations, for instance the first and second, we have an equation without ω, $A^2u\div B^2u=\mathfrak{A}(a-x)\div\mathfrak{B}(b-x)$; and using this to determine x, and then substituting in $\omega^2=A^2u\div\mathfrak{A}(a-x)$, we find

$$\omega^2=\frac{\mathfrak{B}A^2u-\mathfrak{A}B^2u}{(a-b)\mathfrak{A}\mathfrak{B}},$$

where the numerator is a function not in anywise more important than any other linear function of A^2u and B^2u.

38. The function Du vanishes for $u=0$, and we may assume that the corresponding value of x is $=d$. Writing in the other equations $u=0$, they become

$$\begin{aligned} A^20 &= (\alpha^2+\beta^2) = \omega_0^2\mathfrak{A}(a-d),\\ B^20 &= \quad 2\alpha\beta \quad = \omega_0^2\mathfrak{B}(b-d),\\ C^20 &= \ \alpha^2-\beta^2 \ = \omega_0^2\mathfrak{C}(c-d), \end{aligned}$$

where ω_0^2 is what ω^2 becomes on writing therein $x=d$. It is convenient to omit altogether these factors ω^2 and ω_0^2; it being understood that without them, the equations denote not absolute equalities, but only equalities of ratios: thus, without the ω_0^2, the last-mentioned equations would denote $A^20 \ B^20 : C^20=\alpha^2+\beta^2 : 2\alpha\beta : \alpha^2-\beta^2$, $=\mathfrak{A}(a-d) : \mathfrak{B}(b-d) : \mathfrak{C}(c-d)$. The quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ only present themselves in the products $\mathfrak{A}\omega^2$, &c., and their absolute magnitudes are therefore essentially indeterminate, but regarding ω^2 as containing a constant factor of properly determined value, the absolute values of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ may be regarded as determinate, and this is accordingly done in the formulæ $\mathfrak{A}^2=-\mathrm{agh}$, &c., which follow.

Relations between the constants.

39. The formulæ contain the differences of the quantities a, b, c, d; denoting these differences in the usual manner

$$b-c,\ c-a,\ a-b,\ a-d,\ b-d,\ c-d$$

by

$$\mathrm{a},\quad \mathrm{b},\quad \mathrm{c},\quad \mathrm{f},\quad \mathrm{g},\quad \mathrm{h} \qquad .$$

so that

$$\begin{array}{rrrrl} . & -\mathrm{h} & +\mathrm{g} & -\mathrm{a} & =0,\\ \mathrm{h} & . & -\mathrm{f} & -\mathrm{b} & =0,\\ -\mathrm{g} & +\mathrm{f} & . & -\mathrm{c} & =0,\\ \mathrm{a} & +\mathrm{b} & +\mathrm{c} & . & =0, \end{array}$$

and also

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch} \qquad =0,$$

and then assuming the absolute value of one of the quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, we have the system of relations

$$\begin{array}{llll}
\mathfrak{A}^2=-agh, & \mathfrak{B}\mathfrak{C}a=\ \mathfrak{A}\mathfrak{D}f, & \mathfrak{A}bcf=-\mathfrak{B}\mathfrak{C}\mathfrak{D}, & \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D}=abcfgh, \\
\mathfrak{B}^2=\ \ bhf, & \mathfrak{C}\mathfrak{A}b=-\mathfrak{B}\mathfrak{D}g, & \mathfrak{B}cag=\ \mathfrak{C}\mathfrak{A}\mathfrak{D}, & \\
\mathfrak{C}^2=\ \ cfg, & \mathfrak{A}\mathfrak{B}c=-\mathfrak{C}\mathfrak{D}h, & \mathfrak{C}abh=\ \mathfrak{A}\mathfrak{B}\mathfrak{D}, & \\
\mathfrak{D}^2=-abc, & & \mathfrak{D}fgh=-\mathfrak{A}\mathfrak{B}\mathfrak{C}, &
\end{array}$$

$$\begin{array}{llll}
. & c^2\mathfrak{B}^2+b^2\mathfrak{C}^2-f^2\mathfrak{D}^2 & =bcf\,(af+bg+ch), & =0, \\
-c^2\mathfrak{A}^2 & .\ \ +a^2\mathfrak{C}^2-g^2\mathfrak{D}^2 & =cag(\quad ,, \quad), & =0, \\
-b^2\mathfrak{A}^2+a^2\mathfrak{B}^2 & .\ \ -h^2\mathfrak{D}^2 & =abh(\quad ,, \quad), & =0, \\
-f^2\mathfrak{A}^2+g^2\mathfrak{B}^2+h^2\mathfrak{C}^2 & . & =fgh\,(\quad ,, \quad), & =0.
\end{array}$$

It is to be remarked that taking c, a, b, d in the order of decreasing magnitude we have $-$a, b, c, f, g, h all positive; hence $\mathfrak{A}^2$, $\mathfrak{B}^2$ $\mathfrak{C}^2$, $\mathfrak{D}^2$ all real; and taking as we may do, $\mathfrak{D}$ negative, then $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ may be taken positive; that is we have $-$a, b, c, f, g, h, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $-\mathfrak{D}$ all of them positive.

40. We have

$$\begin{aligned}
A^2 0&=\alpha^2+\beta^2=\mathfrak{A}f,\\
B^2 0&=\ 2\alpha\beta\ =\mathfrak{B}g,\\
D^2 0&=\alpha^2-\beta^2=\mathfrak{C}h.
\end{aligned}$$

The foregoing equations

$$k=\frac{B^2 0}{A^2 0},\qquad k'=\frac{C^2 0}{A^2 0},$$

give

$$k=\frac{\mathfrak{B}g}{\mathfrak{A}f},\qquad k'=\frac{\mathfrak{C}h}{\mathfrak{A}f},$$

and we thence have

$$k^2=\frac{bg}{-af},\qquad k'^2=\frac{ch}{-af},\quad \text{satisfying } k^2+k'^2=1.$$

41. Observe further that substituting for a, b, c, f, g, h their values, we have

$$\begin{aligned}
\mathfrak{A}^2&=c-b.b-d.c-d, &&= \ c-d.d-b.b-c,\\
\mathfrak{B}^2&=c-a.c-d.a-d, &&= \ d-a.a-c.c-d,\\
\mathfrak{C}^2&=a-b.a-d.b-d, &&=-.a-b.b-d.d-a,\\
\mathfrak{D}^2&=c-b\,c-a.a-b, &&=-.b-c.c-a.a-b,
\end{aligned}$$

where in the first set of values all the differences are positive, but in the second

set of values, we take the triads of *abcd*, in the cyclical order *bcd*, *cda*, *abc*. There is in this last form an apparent want of symmetry as to the signs (viz. the order which might have been expected is $+-+-$), but taking the order of the letters to be $\mathfrak{C},\mathfrak{A},\mathfrak{B},\mathfrak{D}$ and c,a,b,d, then the cyclical arrangement is

$$\begin{aligned}
\mathfrak{C}^2 &= -.b-d.d-a.a-b.\\
\mathfrak{A}^2 &= -\,d-c.c\,-b.b-d\\
\mathfrak{B}^2 &= -\,c\,-a.a-d\,d-c.\\
\mathfrak{D}^2 &= -\,a-b\,b\,-c.c\,-a
\end{aligned}$$

where the four outside signs are all $-$. Observe that the triads of *abcd*, and *abdc*, are

$$\begin{array}{lllll}
 & bcd, & cda, & dab, & abc,\\
\text{and} & bdc, & dca, & cab, & abd,
\end{array}$$

where in the first and second columns the terms of the same column correspond to each other with a reversal of sign, whereas in the third and fourth columns the lower term of either column corresponds to the upper term of the other column, but without a reversal of sign.

The product-sets, $u\pm u'$: and u' indefinitely small, differential formulæ

42. Coming now to the product-sets, these may be written

$$\begin{array}{l|l}
\tfrac{1}{2}\{\overset{u+u'}{\mathrm{C}}.\overset{u-u'}{\mathrm{A}} + \overset{u+u'}{\mathrm{A}}.\overset{u-u'}{\mathrm{C}}\}=\mathrm{X}_{,}\mathrm{X}_{,}', & \tfrac{1}{2}\{\overset{u+u'}{\mathrm{C}}.\overset{u-u'}{\mathrm{A}} - \overset{u+u'}{\mathrm{A}}.\overset{u-u'}{\mathrm{C}}\}=\mathrm{Y}_{,}\mathrm{Y}_{,}',\\
\;,,\{\mathrm{D.B} + \mathrm{B.D}\}=\mathrm{Y}_{,}\mathrm{X}_{,}', & \;,,\{\mathrm{D.B} - \mathrm{B.D}\}=\mathrm{X}_{,}\mathrm{Y}_{,}',\\
\tfrac{1}{2}\{\mathrm{B.A} + \mathrm{A.B}\}= (\mathrm{P}+\mathrm{Q})(\mathrm{P}'+\mathrm{Q}'), & \tfrac{1}{2}\{\mathrm{B.A} - \mathrm{A.B}\}= (\mathrm{P}-\mathrm{Q})(\mathrm{P}'-\mathrm{Q}'),\\
\;,,\{\mathrm{D.C} + \mathrm{C.D}\}= i(\mathrm{P}-\mathrm{Q})(\mathrm{P}'+\mathrm{Q}'), & \;,,\{\mathrm{D.C} - \mathrm{C.D}\}= i(\mathrm{P}+\mathrm{Q})(\mathrm{P}'-\mathrm{Q}'),\\
\tfrac{1}{2}\{\mathrm{D.A} + \mathrm{A.D}\}= (\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'+i\mathrm{Q}_{,}'), & \tfrac{1}{2}\{\mathrm{D.A} - \mathrm{A.D}\}= (\mathrm{P}_{,}+i\mathrm{Q}_{,})(\mathrm{P}_{,}'-i\mathrm{Q}_{,}'),\\
\;,,\{\mathrm{B.C} + \mathrm{C.B}\}=-i(\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'+i\mathrm{Q}_{,}'), & \;,,\{\mathrm{B.C} - \mathrm{C.B}\}=-i(\mathrm{P}_{,}-i\mathrm{Q}_{,})(\mathrm{P}_{,}'-i\mathrm{Q}_{,}').
\end{array}$$

43. We can from each set form two fractions (each of them a function of $u+u'$ and $u-u'$), which are equal to one and the same function of u' only: for instance, from the first set we have two fractions, each $\dfrac{\mathrm{Y}_{,}'}{\mathrm{X}_{,}'}$. putting in such equation $u=0$, we obtain a new expression for the function of u' involving only the theta-functions $\mathrm{A}u'$, &c.,

which new expression we may then substitute in the equations first obtained · we thus arrive at the six equations

$$\frac{\overset{u+u'}{\mathrm{C}}\ \overset{u-u'}{\mathrm{A}} - \overset{u+u'}{\mathrm{A}}\ \overset{u-u'}{\mathrm{C}}}{\mathrm{D}\ \mathrm{B} + \mathrm{B}\ \mathrm{D}} = \frac{\overset{u+u'}{\mathrm{D}}\ \overset{u-u'}{\mathrm{B}} - \overset{u+u'}{\mathrm{B}}\ \overset{u-u'}{\mathrm{D}}}{\mathrm{C}\ \mathrm{A} + \mathrm{A}\ \mathrm{C}} = \frac{\mathrm{D}u'\,\mathrm{B}u'}{\mathrm{C}u'\,\mathrm{A}u'},$$

$$-\frac{\mathrm{B}\ \mathrm{A} - \mathrm{A}\ .\ \mathrm{B}}{\mathrm{D}\ \mathrm{C} + \mathrm{C}\ \mathrm{D}} = \frac{\mathrm{D}\ \mathrm{C} - \mathrm{C}\ \mathrm{D}}{\mathrm{B}\ \mathrm{A} + \mathrm{A}\ \mathrm{B}} = \frac{\mathrm{D}u'\,\mathrm{C}u}{\mathrm{B}u'\,\mathrm{A}u},$$

$$-\frac{\mathrm{B}\ \mathrm{C} - \mathrm{C}\ \mathrm{B}}{\mathrm{D}\ \mathrm{A} + \mathrm{A}\ \mathrm{D}} = \frac{\mathrm{D}\ \mathrm{A} - \mathrm{A}\ \mathrm{D}}{\mathrm{B}\ \mathrm{C} + \mathrm{C}\ \mathrm{B}} = \frac{\mathrm{D}u\,\mathrm{A}u}{\mathrm{B}u\,\mathrm{C}u},$$

where observe that the expressions all vanish for $u'=0$.

44. Taking herein u' indefinitely small we obtain

$$\frac{\mathrm{A}u\,\mathrm{C}'u - \mathrm{C}u\,\mathrm{A}'u}{\mathrm{B}u\,\mathrm{D}u} = \frac{\mathrm{B}u\,\mathrm{D}'u - \mathrm{D}u.\mathrm{B}'u}{\mathrm{C}u\,\mathrm{A}u} = \frac{\mathrm{D}'0\,\mathrm{B}0}{\mathrm{C}'0\,\mathrm{A}0} = -\mathrm{K}\frac{\mathrm{B}^2 0}{\mathrm{A}^2 0},$$

$$-\frac{\mathrm{A}u\,\mathrm{B}'u - \mathrm{B}u\,\mathrm{A}'u}{\mathrm{C}u\,\mathrm{D}u} = \frac{\mathrm{C}u\,\mathrm{D}'u - \mathrm{D}u\,\mathrm{C}'u}{\mathrm{C}u\,\mathrm{B}u} = \frac{\mathrm{D}'0\,\mathrm{C}'0}{\mathrm{A}0\,\mathrm{B}0} = -\mathrm{K}\frac{\mathrm{C}^2 0}{\mathrm{A}^2 0},$$

$$-\frac{\mathrm{C}u\,\mathrm{B}'u - \mathrm{B}u\,\mathrm{C}'u}{\mathrm{A}u\,\mathrm{D}u} = \frac{\mathrm{A}u\,\mathrm{D}'u - \mathrm{D}u\,\mathrm{A}'u}{\mathrm{B}u\,\mathrm{C}u} = \frac{\mathrm{D}'0\,\mathrm{A}0}{\mathrm{B}0\,\mathrm{C}'0} = -\mathrm{K},$$

where the last column is added in order to introduce K in place of D′0.

45. These formulæ in effect give the derivatives of the quotient-functions in terms of quotient-functions for instance, one of the equations is

$$\frac{d}{du}\frac{\mathrm{D}u}{\mathrm{A}u} = -\mathrm{K}\frac{\mathrm{B}u}{\mathrm{A}u}\cdot\frac{\mathrm{C}u}{\mathrm{A}u},$$

substituting herein for the quotient-fractions their values in terms of x, this becomes

$$\frac{d}{du}\sqrt{\frac{d-x}{a-x}} = -\mathrm{K}\sqrt{\frac{\mathfrak{BC}}{\mathfrak{AD}}}\,\frac{\sqrt{b-x\,.\,c-x}}{a-x}, = -\mathrm{K}\sqrt{\frac{\mathfrak{f}}{\mathfrak{a}}}\,\frac{\sqrt{b-x\,.\,c-x}}{a-x},$$

or the left hand being $= \dfrac{-\tfrac{1}{2}\mathfrak{f}}{(a-x)^{\frac{3}{2}}\sqrt{d-x}}\dfrac{dx}{du}$, this is

$$\mathrm{K}du = \frac{\tfrac{1}{2}\sqrt{\mathfrak{af}}\,dx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}},$$

where on the right hand side it would be better to write $\sqrt{-\mathfrak{af}}$ in the numerator, and $x-d$ in place of $d-x$ in the denominator.

Comparison with JACOBI.

46. The comparison of the formulæ with JACOBI gives

$$\begin{aligned}
\operatorname{sn}\mathrm{K}u &= -\frac{1}{\sqrt{k}}\mathrm{D}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{d-x}{c-x}}\left(\text{or better }\sqrt{\frac{-\mathrm{a}}{\mathrm{g}}}\sqrt{\frac{x-d}{c-x}}\right),\\
\operatorname{cn}\mathrm{K}u &= \sqrt{\frac{k'}{k}}\mathrm{B}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{h}}{\mathrm{g}}}\sqrt{\frac{b-x}{c-x}},\\
\operatorname{dn}\mathrm{K}u &= \sqrt{k'}\mathrm{A}u \div \mathrm{C}u, &&= \sqrt{\frac{\mathrm{h}}{\mathrm{f}}}\sqrt{\frac{a-x}{c-x}},
\end{aligned}$$

where it will be recollected that

$$k^2 = \frac{\mathrm{bg}}{-\mathrm{af}},\quad k'^2 = \frac{\mathrm{ch}}{-\mathrm{af}}.$$

It may be remarked that we seek to determine everything in terms of a, b, c, d. The formula just written down, $k^2 = \mathrm{bg} \div -\mathrm{af}$, gives k in terms of these quantities; and k, K being each given in terms of q, we have virtually K as a function of k, that is of a, b, c, d: but it would not be easy from the expressions of k, K each in terms of q, to deduce the actual expression $\mathrm{K} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$, of K as a function of k.

The square-set, $u \pm u'$.

47. Reverting to the square-set

$$\begin{aligned}
\mathrm{A}(u+u')\mathrm{A}(u-u') &= \mathrm{XX'}+\mathrm{YY'},\\
\mathrm{B}(u+u')\mathrm{B}(u-u') &= \mathrm{YX'}+\mathrm{XY'},\\
\mathrm{C}(u+u')\mathrm{C}(u-u') &= \mathrm{XX'}-\mathrm{YY'},\\
\mathrm{D}(u+u')\mathrm{D}(u-u') &= -\mathrm{YX'}+\mathrm{XY'},
\end{aligned}$$

if we first write herein $u'=0$, and then $u=0$, using the results to determine the values of X, Y, X′, Y′ we find

$$\begin{aligned}
\alpha\mathrm{C}^2u-\beta\mathrm{D}^2u &= (\alpha^2-\beta^2)\mathrm{X}, & \quad \alpha\mathrm{C}^2u'-\beta\mathrm{D}^2u' &= (\alpha^2-\beta^2)\mathrm{X'},\\
\beta\mathrm{C}^2u-\alpha\mathrm{D}^2u &= \quad ,, \quad \mathrm{Y}, & \quad \beta\mathrm{C}^2u'-\alpha\mathrm{D}^2u' &= \quad ,, \quad \mathrm{Y'},
\end{aligned}$$

and thence

$$(\alpha^2-\beta^2)^2\mathrm{XX}'=\alpha^2.\mathrm{C}^2u\mathrm{C}^2u'+\beta^2.\mathrm{D}^2u\mathrm{D}^2u'-\alpha\beta(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),$$
$$,,\quad \mathrm{YY}'=\beta^2 \quad ,, \quad +\alpha^2. \quad ,, \quad -\alpha\beta \quad ,, \quad ,$$

whence

$$(\alpha^2-\beta^2)^2(\mathrm{XX}'+\mathrm{YY}')=(\alpha^2+\beta^2)(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-2\alpha\beta(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),$$
$$(\alpha^2-\beta^2)\ (\mathrm{XX}'-\mathrm{YY}')=\qquad (\mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u'),$$

(where observe that in taking the difference the right hand side becomes divisible by $\alpha^2-\beta^2$, and therefore in the final result we have on the left hand side the simple factor $\alpha^2-\beta^2$ instead of $(\alpha^2-\beta^2)^2$).

Similarly

$$(\alpha^2-\beta^2)\mathrm{YX}'=\alpha\beta(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\alpha^2\mathrm{D}^2u\mathrm{C}^2u'-\beta^2\mathrm{C}^2u\mathrm{D}^2u',$$
$$,,\quad \mathrm{XY}'=\alpha\beta \quad ,, \quad -\beta^2 \quad ,, \quad -\alpha^2 \quad ,, \quad ,$$

and thence

$$(\alpha^2-\beta^2)^2(\ \ \mathrm{YX}'+\mathrm{XY}')\ =2\alpha\beta(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-(\alpha^2+\beta^2)(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),$$
$$(\alpha^2-\beta^2)\ (-\mathrm{YX}'+\mathrm{XY}')\ =\ \mathrm{D}^2u\mathrm{C}^2u'-\mathrm{C}^2u\mathrm{D}^2u',$$

48. Hence recollecting that

$$\mathrm{A}^20=\alpha^2+\beta^2,$$
$$\mathrm{B}^20=2\alpha\beta,$$
$$\mathrm{C}^20=\alpha^2-\beta^2,$$

the original equations become

$$\mathrm{C}^40.\mathrm{A}(u+u')\mathrm{A}(u-u')=\mathrm{A}^20(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\mathrm{B}^20(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),$$
$$\mathrm{C}^40.\mathrm{B}(u+u')\mathrm{B}(u-u')=\mathrm{B}^20(\mathrm{C}^2u\mathrm{C}^2u'+\mathrm{D}^2u\mathrm{D}^2u')-\mathrm{A}^20(\mathrm{C}^2u\mathrm{D}^2u'+\mathrm{D}^2u\mathrm{C}^2u'),$$
$$\mathrm{C}^20.\mathrm{C}(u+u')\mathrm{C}(u-u')=\qquad \mathrm{C}^2u\mathrm{C}^2u'-\mathrm{D}^2u\mathrm{D}^2u',$$
$$\mathrm{C}^20.\mathrm{D}(u+u')\mathrm{D}(u-u')=\qquad \mathrm{D}^2u\mathrm{C}^2u'-\mathrm{C}^2u\mathrm{D}^2u'.$$

49. It will be observed that the four products $\mathrm{A}(u+u')\mathrm{A}(u-u')$, &c., are each of them expressed in terms of C^2u, D^2u, C^2u', D^2u'. Since each of the squared functions A^2u, B^2u, C^2u, D^2u is a linear function of any two of them, and the like as regards A^2u', B^2u',C^2u', D^2u', it is clear that in each equation we can on the right hand side introduce any two at pleasure of the squared functions of u, and any two at pleasure of the squared functions of u'. But all the forms so obtained are of course identical if, taking x' the same function of u' that x is of u, we introduce on the right hand side x, x' instead of u, u'; and the values of $\mathrm{A}(u+u').\mathrm{A}(u-u')$ are found to be equal to multiples of ∇, ∇_1, ∇_2, ∇_3, where

$$\nabla = x-x' \quad \nabla_1 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & a+d, & ad \\ 1, & b+c, & bc \end{vmatrix}, \quad \nabla_2 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & b+d, & bd \\ 1, & c+d, & cd \end{vmatrix}, \quad \nabla_3 = \begin{vmatrix} 1, & x+x', & xx' \\ 1, & c+d, & cd \\ 1, & a+b, & ab \end{vmatrix}.$$

50. In fact, from the equations $A^2u = \mathfrak{A}(a-x)$, $A^2u' = \mathfrak{A}(a-x')$ we have

$$\nabla = \frac{1}{\mathrm{a}\mathfrak{B}\mathfrak{C}}(B^2uC^2u' - C^2uB^2u'), \quad = \frac{1}{\mathrm{b}\mathfrak{C}\mathfrak{A}}(C^2uA^2u' - A^2uC^2u'), \quad = \frac{1}{\mathrm{c}\mathfrak{A}\mathfrak{B}}(A^2uB^2u' - B^2uA^2u'),$$

$$= \frac{1}{\mathrm{f}\mathfrak{A}\mathfrak{D}}(A^2uD^2u' - D^2uA^2u'), \quad = \frac{1}{\mathrm{g}\mathfrak{B}\mathfrak{D}}(B^2uD^2u' - D^2uB^2u'), \quad = \frac{1}{\mathrm{h}\mathfrak{C}\mathfrak{D}}(C^2uD^2u' - D^2uC^2u'),$$

where it will be recollected that $\mathrm{f}\mathfrak{A}\mathfrak{D} = \mathrm{a}\mathfrak{B}\mathfrak{C}$, $-\mathrm{g}\mathfrak{B}\mathfrak{D} = \mathrm{b}\mathfrak{C}\mathfrak{A}$, $-\mathrm{h}\mathfrak{C}\mathfrak{D} = \mathrm{c}\mathfrak{A}\mathfrak{B}$. Moreover

$$(b-c)\nabla_1 = -\begin{vmatrix} b-x.b-x', & c-x.c-x' \\ b-a.b-d, & c-a.c-d \end{vmatrix}, \quad (a-d)\nabla_1 = \begin{vmatrix} a-x.a-x', & d-x.d-x' \\ a-b.a-c, & d-b.d-c \end{vmatrix},$$

$$(c-a)\nabla_2 = -\begin{vmatrix} c-x.c-x', & a-x.a-x' \\ c-b.c-d, & a-b.a-d \end{vmatrix}, \quad (b-d)\nabla_2 = \begin{vmatrix} b-x.b-x', & d-x.d-x' \\ b-c.b-a, & d-c\ b-a \end{vmatrix},$$

$$(a-b)\nabla_3 = -\begin{vmatrix} a-x.a-x', & b-x.b-x' \\ a-c.a-d, & b-c.b-d \end{vmatrix}, \quad (c-d)\nabla_3 = \begin{vmatrix} c-x.c-x', & d-x.d-x' \\ c-a.c-d, & d-a.d-b \end{vmatrix},$$

or as these may be written

$$\nabla_1 = -\frac{1}{\mathrm{a}}\{\mathrm{bh}.\,b-x.b-x'. + \mathrm{cg}.\,c-x.c-x'\}, \quad = \frac{1}{\mathrm{f}}\{\mathrm{gh}.\,a-x.a-x'. + \mathrm{bc}.\,d-x.d-x'\},$$

$$\nabla_2 = -\frac{1}{\mathrm{b}}\{\mathrm{cf}.\,c-x.c-x'. + \mathrm{ah}.\,a-x.a-x'\}, \quad = \frac{1}{\mathrm{g}}\{\mathrm{hf}.\,b-x.b-x'. + \mathrm{ca}.\,d-x.d-x'\},$$

$$\nabla_3 = -\frac{1}{\mathrm{c}}\{\mathrm{ag}.\,a-x.a-x'. + \mathrm{bf}.\,b-x.b-x'\}, = \frac{1}{\mathrm{h}}\{\mathrm{fg}.\,c-x.c-x'. + \mathrm{ab}.\,d-x.d-x'\},$$

that is

$$\nabla_1 = -\frac{1}{\mathrm{a}}\left\{\frac{\mathrm{bh}}{\mathfrak{B}^2}B^2uB^2u' + \frac{\mathrm{cg}}{\mathfrak{C}^2}C^2uC^2u'\right\}, \quad = {}^{1}\left\{\frac{\mathrm{gh}}{\mathfrak{A}^2}A^2uA^2u' + \frac{\mathrm{hc}}{\mathfrak{D}^2}D^2uD^2u'\right\},$$

$$\nabla_2 = -\frac{1}{\mathrm{b}}\left\{\frac{\mathrm{cf}}{\mathfrak{C}^2}C^2uC^2u' + \frac{\mathrm{ah}}{\mathfrak{A}^2}A^2uA^2u'\right\}, \quad = \frac{1}{\mathrm{g}}\left\{\frac{\mathrm{hf}}{\mathfrak{B}^2}B^2uB^2u' + \frac{\mathrm{ca}}{\mathfrak{D}^2}D^2uD^2u'\right\},$$

$$\nabla_3 = -\frac{1}{\mathrm{c}}\left\{\frac{\mathrm{ag}}{\mathfrak{A}^2}A^2uA^2u' + \frac{\mathrm{bf}}{\mathfrak{C}^2}B^2uB^2u'\right\}, \quad = \frac{1}{\mathrm{h}}\left\{\frac{\mathrm{fg}}{\mathfrak{C}^2}C^2uC^2u' + \frac{\mathrm{ab}}{\mathfrak{D}^2}D^2uD^2u'\right\},$$

or finally

$$\nabla_1 = -\frac{1}{\mathrm{af}}(\quad B^2uB^2u' + C^2uC^2u'), = -\frac{1}{\mathrm{af}}(A^2uA^2u' + D^2uD^2u'),$$

$$\nabla_2 = -\frac{1}{\mathrm{bg}}(\quad C^2uC^2u' - A^2uA^2u'), = \quad \frac{1}{\mathrm{bg}}(B^2uB^2u' - D^2uD^2u'),$$

$$\nabla_3 = -\frac{1}{\mathrm{ch}}(-A^2uA^2u' + B^2uB^2u'), = \quad \frac{1}{\mathrm{ch}}(C^2uC^2u' - D^2uD^2u').$$

51. Hence ∇, ∇_1, ∇_2, ∇_3 denoting these functions of x, x' or of u, u', we have

$$A(u+u')A(u-u') = \frac{\mathfrak{A}}{\mathrm{gh}}\nabla_1,$$

$$B(u+u')B(u-u') = \frac{\mathfrak{B}}{\mathrm{hf}}\nabla_2,$$

$$C(u+u')C(u-u') = \frac{\mathfrak{C}}{\mathrm{fg}}\nabla_3,$$

$$D(u+u')D(u-u') = \mathfrak{D}\nabla.$$

The square-set $u \pm u'$, u' indefinitely small: differential formulæ of the second order.

52. I consider the original form

$$C^2 0 C(u+u')C(u-u') = C^2uC^2u' - D^2uD^2u',$$

(which is of course included in the last-mentioned equations).

Writing this in the form

$$C^2 0 \frac{C(u+u')C(u-u')}{C^2u} = C^2u' - \frac{D^2uD^2u'}{C^2u},$$

and taking u' indefinitely small, whence

$$C(u+u') = Cu + u'C'u + \tfrac{1}{2}u'^2C''u, \quad Cu' = C0,$$

$$C(u-u') = Cu - u'C'u + \tfrac{1}{2}u'^2C''u, \quad Du' = \qquad u'D'0,$$

$$C(u+u')C(u-u') = C^2u + u'^2\{CuC''u - (C'u)^2\},$$

the equation becomes

$$C^2 0\left(1 + u'^2\left\{\frac{C''u}{Cu} - \left(\frac{C'u}{Cu}\right)^2\right\}\right) = C^2 0 + u'^2\left\{C0C''0 - (D'0)^2\frac{D^2u}{C^2u}\right\},$$

that is

$$\frac{C''u}{Cu} - \left(\frac{C'u}{Cu}\right)^2 = \frac{C''0}{C0} - \left(\frac{D'0}{C0}\right)^2\frac{D^2u}{C^2u},$$

viz., we have $\left(\frac{d}{du}\right)^2 \log \mathrm{C}u$ expressed in terms of the quotient-function $\frac{\mathrm{D}^2u}{\mathrm{C}^2u}$, and consequently $\mathrm{C}u$ given as an exponential, the argument of which depends on the double integral $\int du \int du \frac{\mathrm{D}^2u}{\mathrm{C}^2u}$.

53. To complete the result I write the equation in the form

$$\frac{d^2}{du^2}\log \mathrm{C}u = \frac{\mathrm{C}''0}{\mathrm{C}0} - \frac{1}{l}\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2 + \frac{1}{l}\left(\frac{\mathrm{D}'0}{\mathrm{C}0}\right)^2\left(1 - k\,\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\right);$$

$\frac{\mathrm{D}'0}{\mathrm{C}'0}$ is $= -\sqrt{l}\,\mathrm{K}$, and $\frac{\mathrm{C}''0}{\mathrm{C}0}$ is $= \mathrm{K}(\mathrm{K}-\mathrm{E})$; hence the equation is

$$\frac{d^2}{du^2}\log \mathrm{C}u = \mathrm{K}^2\left(1 - \frac{\mathrm{E}}{\mathrm{K}} - k\,\frac{\mathrm{D}^2u}{\mathrm{C}^2u}\right),\ = \mathrm{K}^2\left(1 - \frac{\mathrm{E}}{\mathrm{K}} - k^2\mathrm{sn}^2\mathrm{K}u\right),$$

or integrating twice, observing that $\frac{d}{du}\log \mathrm{C}u$ and $\log \mathrm{C}u$, for $u=0$, become $=0$ and $\log \mathrm{C}0$ respectively,

$$\log \mathrm{C}u = \log \mathrm{C}0 + \tfrac{1}{2}\left(1 - \frac{\mathrm{E}}{\mathrm{K}}\right)\mathrm{K}^2u^2 - k^2\int_0 du \int_0 du\,\mathrm{K}^2\mathrm{sn}^2\mathrm{K}u,$$

which is in fact

$$\log \Theta(\mathrm{K}u) = \log \mathrm{C}0 + \tfrac{1}{2}\left(1 - \frac{\mathrm{E}}{\mathrm{K}}\right)\mathrm{K}^2u^2 - k^2\int_0 du \int_0 du\,\mathrm{K}^2\mathrm{sn}^2\mathrm{K}u,$$

agreeing with JACOBI's

$$\log \Theta u = \log \Theta 0 + \tfrac{1}{2}\left(1 - \frac{\mathrm{E}}{\mathrm{K}}\right)u^2 - k^2\int_0 du \int_0 du\ \mathrm{sn}^2u.$$

Elliptic integrals of the third kind.

54. We may write

$$\frac{\mathrm{A}(u+u')\mathrm{A}(u-u')}{\mathrm{A}^2u\mathrm{A}^2u'} = \frac{1}{\mathfrak{A}\mathrm{gh}}\,\frac{\nabla_1}{a-x.a-x'},$$

$$\frac{\mathrm{B}(u+u')\mathrm{B}(u-u')}{\mathrm{B}^2u\mathrm{B}^2u'} = \frac{1}{\mathfrak{B}\mathrm{hf}}\,\frac{\nabla_2}{b-x.b-x'},$$

$$\frac{\mathrm{C}(u+u')\mathrm{C}(u-u')}{\mathrm{C}^2u\mathrm{C}^2u'} = \frac{1}{\mathfrak{C}\mathrm{fg}}\,\frac{\nabla_3}{c-x.c-x'},$$

$$\frac{\mathrm{D}(u+u')\mathrm{D}(u-u')}{\mathrm{D}^2u\mathrm{D}^2u'} = \frac{1}{\mathfrak{D}}\,\frac{x-x'}{d-x.d-x'}.$$

We differentiate logarithmically in regard to u'. Observing that

$$\mathrm{K}du'=\frac{\frac{1}{2}\sqrt{\mathrm{af}}dx'}{\sqrt{a-x'.b-x'.c-x'.d-x'}},=\frac{\frac{1}{2}\sqrt{\mathrm{af}}}{\sqrt{\mathrm{X}'}}dx'$$

suppose, the first equation gives

$$\frac{\mathrm{A}'u}{\mathrm{A}u}+\tfrac{1}{2}\frac{\mathrm{A}'(u-u')}{\mathrm{A}(u+u')}-\tfrac{1}{2}\frac{\mathrm{A}'(u+u')}{\mathrm{A}(u+u')}=-\frac{\mathrm{K}\sqrt{\mathrm{X}'}}{\sqrt{\mathrm{af}}}\frac{d}{dx'}\log\frac{\nabla_1}{a-x'},$$

and if for a moment

$$\nabla_1,=\begin{vmatrix}1, & x+x', & xx'\\ 1, & a+d, & ad\\ 1, & b+c, & bc\end{vmatrix}, \text{ is put }=\mathrm{P}(a-x')+\mathrm{Q}(d-x'),$$

then

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'},=\frac{d}{dx'}\log\left(\mathrm{P}+\mathrm{Q}\frac{d-x'}{a-x'}\right)\text{ is }=\frac{\mathrm{Q}(d-a)}{(a-x')\nabla_1},=-\frac{\mathrm{Qf}}{(a-x')\nabla_1}.$$

But writing $x'=a$ we have

$$\mathrm{Q}(d-a),=-\mathrm{Qf}=\begin{vmatrix}1, & a+x, & ax\\ 1, & a+d, & ad\end{vmatrix},=(a-b)(a-c)(d-x),=-\mathrm{bc}(d-x),$$

that is, $\mathrm{Qf}=-\mathrm{bc}(d-x)$, or

$$\frac{d}{dx'}\log\frac{\nabla_1}{a-x'}=\frac{\mathrm{bc}(d-x')}{(a-x')\nabla_1}.$$

Hence the equation is

$$2\frac{\mathrm{A}'(u')}{\mathrm{A}(u')}+\frac{\mathrm{A}'(u-u')}{\mathrm{A}(u-u')}-\frac{\mathrm{A}'(u+u')}{\mathrm{A}(u+u')}=\frac{2\mathrm{Kbc}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(a-x')\nabla_1};$$

and similarly

$$2\frac{\mathrm{B}'(u')}{\mathrm{B}(u')}+\frac{\mathrm{B}'(u-u')}{\mathrm{B}(u-u')}-\frac{\mathrm{B}'(u+u')}{\mathrm{B}(u+u')}=\frac{2\mathrm{Kca}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(b-x')\nabla_2},$$

$$2\frac{\mathrm{C}'(u')}{\mathrm{C}(u')}+\frac{\mathrm{C}'(u-u')}{\mathrm{C}(u-u')}-\frac{\mathrm{C}'(u+u')}{\mathrm{C}(u+u')}=\frac{2\mathrm{Kab}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(c-x')\nabla_3},$$

$$2\frac{\mathrm{D}'(u')}{\mathrm{D}(u')}+\frac{\mathrm{D}'(u-u')}{\mathrm{D}(u-u')}-\frac{\mathrm{D}'(u+u')}{\mathrm{D}(u+u')}=\frac{2\mathrm{K}}{\sqrt{\mathrm{af}}}\sqrt{\mathrm{X}'}\frac{d-x}{(d-x')(x-x')}.$$

55. Multiply each of these equations by du, $=\frac{1}{2}\frac{\sqrt{\mathrm{af}}}{\mathrm{K}}\frac{dx}{\sqrt{\mathrm{X}}}$, and integrate. We have equations such as

$$2u\frac{A'(u')}{A(u')}+\log\frac{A(u-u')}{A(u+u')}=\text{const.}+\frac{bc\sqrt{X'}}{\sqrt{af}(a-x')}\int\frac{(d-x)dx}{\nabla_1\sqrt{X}},$$

showing how the integrals of the third kind

$$\int\frac{(d-x)dx}{\nabla_1\sqrt{X}},\quad \int\frac{(d-x)dx}{\nabla_2\sqrt{X}},\quad \int\frac{(d-x)dx}{\nabla_3\sqrt{X}},\quad \int\frac{(d-x)dx}{(x-x')\sqrt{X}}$$

depend on the theta-functions.

If, instead, we work with the original equation

$$C^20\,\frac{C(u+u')C(u-u')}{C^2u.C^2u'}=1-\frac{D^2u}{C^2u}\frac{D^2u'}{C^2u'},$$

we find in the same way

$$2\frac{C'(u')}{C(u')}+\frac{C'(u-u')}{C(u-u')}-\frac{C(u+u')}{C(u+u')}=-\frac{d}{du'}\log\left(1-\frac{D^2uD^2u'}{C^2uC^2u'}\right),$$

$$=-\frac{d}{du'}\log\left(1-k^2\text{sn}^2Ku\ \text{sn}^2Ku'\right),$$

$$=\frac{2k^2K\text{sn}Ku'\text{cn}Ku'\text{dn}Ku'\text{sn}^2Ku}{1-k^2\text{sn}^2Ku'\text{sn}^2Ku};$$

or multiplying by $\frac{1}{2}du$ and integrating

$$u\frac{C'(u')}{C(u')}+\tfrac{1}{2}\log\frac{C(u-u')}{C(u+u')}=\int\frac{k^2\text{sn}Ku'\text{cn}Ku'\text{dn}Ku'\text{sn}^2Ku.Kdu}{1-k^2\text{sn}^2Ku'\text{sn}^2Ku},$$

which is in fact JACOBI's equation

$$u\frac{\Theta'a}{\Theta a}+\tfrac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}=\int\frac{\text{sn}\,a\,\text{cn}\,a\,\text{dn}\,a\,\text{sn}^2u\,du}{1-k^2\text{sn}^2a\,\text{sn}^2u},=\Pi(u,a).$$

I do not effect the operation but consider the forms first obtained,

$$A(u+u')A(u-u')=\frac{\mathfrak{A}}{gh}\nabla_1,\ \&c.,$$

as the equivalent of JACOBI's last-mentioned equation.

Addition-formulæ.

56. The addition-theorem for the quotient-functions is of course given by means of the theorem for the elliptic functions: but more elegantly by the formulæ relating to

the form $dx \div \sqrt{a-x.b-x.c-x.d-x}$ obtained in my paper "On the Double ϑ-Functions" ('Crelle-Borchardt,' tom. 87 (1879), pp. 74–81), viz.: for the differential equation

$$\frac{dx}{\sqrt{\mathrm{X}}}+\frac{dy}{\sqrt{\mathrm{Y}}}-\frac{dz}{\sqrt{\mathrm{Z}}}=0,$$

to obtain the particular integral which for $y=d$ reduces itself to $z=x$, we must, in the formulæ of the paper just referred to, interchange a and d: and writing for shortness $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}=a-x, b-x, c-x, d-x$, and similarly $\mathrm{a}_{,}, \mathrm{b}_{,}, \mathrm{c}_{,}, \mathrm{d}_{,}=a-y, b-y, c-y, d-y$, then when the interchange is made, the formulæ become

$$\sqrt{\frac{d-z}{a-z}}$$

$$=\frac{\sqrt{\overline{d-b}\,\overline{d-c}}\,\{\sqrt{\mathrm{adb_{,}c_{,}}}+\sqrt{\mathrm{a_{,}d_{,}bc}}\}}{(bc,\, ad)},$$

$$=\frac{\sqrt{\overline{d-b}\,\overline{d-c}}\,(x-y)}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{\overline{d-b}\,\overline{d-c}}\,\{\sqrt{\mathrm{bdc_{,}a_{,}}}+\sqrt{\mathrm{b_{,}d_{,}ca}}\}}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{\overline{d-b}\,\overline{d-c}}\,\{\sqrt{\mathrm{cda_{,}b_{,}}}+\sqrt{\mathrm{abc_{,}d_{,}}}\}}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}},$$

$$\sqrt{\frac{b-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}+(b-d)\sqrt{\mathrm{cdc_{,}d_{,}}}\}}{(bc,\, ad)},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{\sqrt{\mathrm{bda_{,}c}}-\sqrt{\mathrm{b_{,}d_{,}ac}}\}}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}(ac,\, bd)}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{\frac{d-b}{d-a}}\{(d-a)\sqrt{\mathrm{bcb_{,}c}}+(b-c)\sqrt{\mathrm{aba_{,}b_{,}}}\}}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}},$$

$$\sqrt{\frac{c-z}{a-z}}$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{(d-b)\sqrt{\mathrm{cac_{,}a_{,}}}+(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}\}}{(bc,\, ad)},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{\sqrt{\mathrm{cda_{,}b_{,}}}-\sqrt{\mathrm{abc_{,}d_{,}}}\}}{\sqrt{\mathrm{adb_{,}c_{,}}}-\sqrt{\mathrm{a_{,}d_{,}bc}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}\{(d-a)\sqrt{\mathrm{bcb_{,}c_{,}}}-(b-c)\sqrt{\mathrm{ada_{,}d_{,}}}\}}{(d-c)\sqrt{\mathrm{aba_{,}b_{,}}}-(b-a)\sqrt{\mathrm{cdc_{,}d_{,}}}},$$

$$=\frac{\sqrt{\frac{d-c}{d-a}}(ab,\, cd)}{(d-b)\sqrt{\mathrm{aca_{,}c_{,}}}-(c-a)\sqrt{\mathrm{bdb_{,}d_{,}}}}.$$

57. In the foregoing formulæ $(bc,\ ad)$ $(ac,\ bd)$ and $(ad,\ bc)$ denote respectively

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix}, \quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & c+a, & ca \\ 1, & b+d, & bd \end{vmatrix}, \quad \begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix};$$

and substituting for $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ their values, and for a, b, &c., writing again $a-x$, $b-x$, &c., we have moreover

$$\begin{array}{ll|ll}
A^2u=\sqrt{\overline{c-b}.\overline{b-d}.\overline{c-d}} & (a-x), & A^2v=\sqrt{\quad ,, \quad} & (a-y), \\
B^2u=\sqrt{\overline{c-a}.\overline{c-d}.\overline{a-d}} & (b-x), & B^2v=\sqrt{\quad ,, \quad} & (b-y), \\
C^2u=\sqrt{\overline{a-b}.\overline{a-d}.\overline{b-d}} & (c-x), & C^2v=\sqrt{\quad ,, \quad} & (c-y), \\
D^2u=\sqrt{\overline{c-b}.\overline{c-a}.\overline{a-b}} & (d-x), & D^2v=\sqrt{\quad ,, \quad} & (d-y),
\end{array}$$

$$\begin{array}{ll}
A^2(u+v)=\sqrt{\quad ,, \quad} & (a-z), \\
B^2(u+v)=\sqrt{\quad ,, \quad} & (b-z), \\
C^2(u+v)=\sqrt{\quad ,, \quad} & (c-z), \\
D^2(u+v)=\sqrt{\quad ,, \quad} & (d-z),
\end{array}$$

the constant multipliers being of course the same in the three columns respectively. According to what precedes, the radical of the fourth line should be taken with the sign $-$. The functions $(bc,\ ad)$, &c., contained in the formulæ require a transformation such as

$$(b-c)\ (bc,\ ad)=\begin{vmatrix} \overline{b-x}.\overline{b-y}, & \overline{c-x}.\overline{c-y} \\ \overline{b-a}.\overline{b-d}, & \overline{c-a}.\overline{c-d} \end{vmatrix}$$

in order to make them separately homogeneous in the differences $a-x$, $b-x$, $c-x$, $d-x$, and $a-y$, $b-y$, $c-y$, $d-y$, and therefore expressible as linear functions of the squared functions A^2u, &c., and A^2v, &c., respectively: and the formulæ then give the quotient-functions $A(u+v)\div D(u+v)$ &c., in terms of the quotient-functions of u and v respectively.

Doubly infinite product-forms.

58. The functions Au, Bu, Cu, Du may be expressed each as a doubly infinite product. Writing for shortness

$$m \quad + \quad n.\frac{a}{\pi i} = (m, n),$$

$$m+1+ \quad n.\frac{a}{\pi i} = (\bar{m}, n),$$

$$m \quad +(n+1)\frac{a}{\pi i} = (m, \bar{n}),$$

$$m+1+(n+1)\frac{a}{\pi i} = (\bar{m}, \bar{n}),$$

then the formulæ are

$$\mathrm{A}u = \mathrm{A}0.\ \Pi\Pi\left\{1+\frac{u}{(\bar{m}, \bar{n})}\right\},$$

$$\mathrm{B}u = \mathrm{B}0.\ \Pi\Pi\left\{1+\frac{u}{(\bar{m}, n)}\right\},$$

$$\mathrm{C}u = \mathrm{C}0.\ \Pi\Pi\left\{1+\frac{u}{(m, \bar{n})}\right\},$$

$$\mathrm{D}u = \mathrm{D}'0.u\Pi\Pi\left\{1+\frac{u}{(m, n)}\right\},$$

where in all the formulæ m, n denote even integers having all values whatever, zero included, from $-\infty$ to $+\infty$; only in the formula for Du, the term for which m and n are simultaneously $=0$, is to be omitted.

59. But a further definition in regard to the limits is required: first, we assume that m has the corresponding positive and negative values, and similarly that n has corresponding positive and negative values*; or say, in the four formulæ respectively, we consider m, n as extending

m from $-\mu$ to $\mu+2$, n from $-\nu$ to $\nu+2$,
" " $-\mu$ " $\mu+2$, " " $-\nu$ " ν,
" " $-\mu$ " μ , " " $-\nu$ " $\nu+2$,
" " $-\mu$ " μ , " " $-\nu$ " ν,

where μ and ν are each of them ultimately infinite. But, secondly, it is necessary that μ should be indefinitely larger than ν, or say that ultimately $\frac{\nu}{\mu}=0$.

60. In fact, transforming the q-series into products as in the 'Fundamenta Nova,' and neglecting for the moment mere constant factors, we have

* This is more than is necessary; it would be enough if the ultimate values of m were $-\mu$, μ', μ and μ' being in a ratio of equality; and the like as regards n. But it is convenient that the numbers should be absolutely equal.

$$\begin{aligned}
\mathrm{A}u &= \qquad (1+2q\cos\pi u+q^2)(1+2q^3\cos\pi u+q^6)\ldots, \\
\mathrm{B}u &= \cos\tfrac{1}{2}\pi u(1+2q^2\cos\pi u+q^4)(1+2q^4\cos\pi u+q^8)\ldots, \\
\mathrm{C}u &= \qquad (1-2q\cos\pi u+q^2)(1-2q^3\cos\pi u+q^6), \\
\mathrm{D}u &= \sin\tfrac{1}{2}\pi u(1-2q^2\cos\pi u+q^4)(1-2q^4\cos\pi u+q^8),
\end{aligned}$$

and writing for a moment $\alpha=\frac{a}{\pi i}$ and therefore $q^{\frac{1}{2}}+q^{-\frac{1}{2}},=e^{\frac{1}{2}\pi\alpha i}+e^{-\frac{1}{2}\pi\alpha i},=2\cos\frac{1}{2}\pi\alpha$, &c., each of these expressions is readily converted into a singly infinite product of sines or cosines

$$\begin{aligned}
\mathrm{A}u &= \Pi.\cos\tfrac{1}{2}\pi(u+\bar{n}\alpha), \\
\mathrm{B}u &= \Pi.\cos\tfrac{1}{2}\pi(u+n\alpha), \\
\mathrm{C}u &= \Pi.\sin\tfrac{1}{2}\pi(u+\bar{n}\alpha), \\
\mathrm{D}u &= \Pi.\sin\tfrac{1}{2}\pi(u+n\alpha),
\end{aligned}$$

where $\bar{n}$ is written to denote $n+1$, and n has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\nu$ to ν, if ν is ultimately infinite.

61. The sines and cosines are converted into infinite products by the ordinary formulæ, which neglecting constant factors may conveniently be written

$$\sin\tfrac{1}{2}\pi u=\Pi(u+m),\ \cos\tfrac{1}{2}\pi u=\Pi(u+\bar{m}),$$

where $\bar{m}$ is written to denote $m+1$, and m has all positive or negative even values (zero included) from $-\infty$ to $+\infty$, or more accurately from $-\mu$ to μ, if μ be ultimately infinite. But in applying these formulæ to the case of a function such as $\sin\frac{1}{2}\pi(u+n\alpha)$, it is assumed that the limiting values $\mu, -\mu$ of m are indefinitely large in regard to $u+n\alpha$; and therefore, since n may approach to its limiting value $\pm\nu$, it is necessary that μ should be indefinitely large in comparison with ν, or that $\frac{\nu}{\mu}=0$.

62. It is on account of this unsymmetry as to the limits $\frac{\nu}{\mu}=0, \frac{\mu}{\nu}=\infty$, that we have 1 as a quarter period, $\frac{a}{\pi i}$ only as a quarter-quasi-period of the single theta-functions.

The transformation q to r, $\log q \log r=\pi^2$.

63. In general, if we consider the ratio of two such infinite products where for the first the limits are $(\pm\mu, \pm\nu)$, and for the second they are $(\pm\mu', \pm\nu')$, and where for convenience we take $\mu>\mu'$, $\nu>\nu'$, then the quotient, say $[\mu, \nu]\div[\mu', \nu]$ is $=\exp.(\mathrm{M}u^2)$ where

$$M = -\tfrac{1}{8}\iint \frac{dmdn}{(m,n)^2}$$

taken over the area included between the two rectangles. We have $(m, n) = m + \frac{a}{\pi i}n$, $= m + i\theta n$ suppose, where (a being negative) $\theta = -\frac{a}{\pi}$, is positive: the integral is

$$\begin{aligned}
\iint \frac{dmdn}{(m+i\theta n)^2}, &= \int dm.\, -\frac{1}{i\theta}\left(\frac{1}{m+i\theta n}\right)_{-\nu}^{\nu}, \\
&= \frac{1}{i\theta}\int dm\left(\frac{1}{m-i\theta\nu} - \frac{1}{m+i\theta\nu}\right), \\
&= \frac{1}{i\theta}\log\frac{m-i\theta\nu}{m+i\theta\nu};
\end{aligned}$$

or finally between the proper limits the value is

$$= \frac{2}{i\theta}\left\{\log\left(\frac{\mu-i\theta\nu}{\mu+i\theta\nu}\right) - \log\left(\frac{\mu'-i\theta\nu'}{\mu'+i\theta\nu'}\right)\right\},$$

where the logarithms are $\log(\mu - i\theta\nu) = \log\sqrt{\mu^2+\nu^2} - i\tan^{-1}\frac{\theta\nu}{\mu}$, &c., and the $\tan^{-1}$ denotes always an arc between the limits $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$. Hence if $\frac{\mu}{\nu} = \infty$, $\frac{\mu'}{\nu'} = 0$, the value is $\frac{2}{i\theta}(-0i - 0i + \frac{1}{2}\pi i + \frac{1}{2}\pi i) = \frac{2\pi}{\theta}, = -\frac{2\pi^2}{a}$; or $K = \frac{1}{4}\frac{\pi^2}{a}$. Hence finally

$$[\mu \div \nu, = \infty] \div [\mu \div \nu, = 0] = \exp\left(\tfrac{1}{4}\frac{\pi^2}{a}u^2\right).$$

64. We have $a, = \log q$, negative; hence taking r such that $\log q \log r = \pi^2$, we have $a' = \log r$, also negative; and r, like q, is positive and less than 1. We consider the theta-functions which depend on r in the same manner that the original functions did on q, say these are

$$A(u, r) = A\,(0, r)\ \Pi\Pi\left\{1 + \frac{u}{\bar{m} + n\frac{a'}{\pi i}}\right\},$$

$$B(u, r) = B\,(0, r)\ \Pi\Pi\left\{1 + \frac{u}{m + n\frac{a'}{\pi i}}\right\},$$

$$C(u, r) = C\,(0, r)\ \Pi\Pi\left\{1 + \frac{u}{m + n\frac{a'}{\pi i}}\right\},$$

$$D(u, r) = D'(0, r)u\Pi\Pi\left\{1 + \frac{u}{m + n\frac{a'}{\pi i}}\right\},$$

limits as before, and in particular $\frac{\mu}{\nu}=\infty$; it is at once seen that if in the original functions, which I now call $A(u, q)$, $B(u, q)$, $C(u, q)$, $D(u, q)$, we write $\frac{au}{\pi i}$ for u, we obtain the same infinite products which present themselves in the expressions of the new functions $A(u, r)$, &c., only with a different condition as to the limits; we have for instance

$$\Pi\Pi\left(1+\frac{\frac{au}{\pi i}}{m+n\frac{a}{\pi i}}\right)=\Pi\Pi\left(1+\frac{u}{n-m\frac{a'}{\pi i}}\right),=\Pi\Pi\left(1+\frac{u}{n+m\frac{a'}{\pi i}}\right),$$

which, interchanging m, n, and of course also μ, ν, is

$$=\Pi\Pi\left(1+\frac{u}{m+n\frac{a'}{\pi i}}\right),$$

with the condition $\frac{\mu}{\nu}=0$ instead of $\frac{\mu}{\nu}=\infty$. Hence disregarding for the moment constant factors, and observing that a is replaced by a', we have

$$D(u, r)\div D\left(\frac{au}{\pi i}, q\right)=[\mu\div\nu,=\infty]\div[\mu\div\nu,=0]$$
$$=\exp\left(\tfrac{1}{4}\frac{\pi^2}{a'}u^2\right), =\exp\left(\tfrac{1}{4}u^2\log q\right).$$

65. We have equations of this form for the four functions, but with a proper constant multiplier in each equation: the equations, in fact, are

$$A(u, r)=\{A(0, r)\div A(0, q)\}\ \exp\left(\tfrac{1}{4}u^2\log q\right)A\left(\frac{au}{\pi i}, q\right),$$

$$B(u, r)=\{B(0, r)\div B(0, q)\}\qquad ,,\qquad B\left(\frac{au}{\pi i}, q\right),$$

$$C(u, r)=\{C(0, r)\div C(0, q)\}\qquad ,,\qquad C\left(\frac{au}{\pi i}, q\right),$$

$$D(u, r)=\{D'(0, r)\div D'(0, q)\}\frac{\pi i}{a}\qquad ,,\qquad D\left(\frac{au}{\pi i}, q\right).$$

It is to be observed that r is the same function of k' that q is of k: this would at once follow from JACOBI's equation $\log q=-\frac{\pi K'}{K}$, for then $\log q\log r=\pi^2$ and therefore $\log r=-\frac{\pi K'}{K}$ $\left(\text{only we are not at liberty to use the relation in question }\log q=-\frac{\pi K'}{K}\right)$, and assuming it to be true we have

$$k=\frac{B^2(0,q)}{A^2(0,q)},\ k'=\frac{C^2(0,q)}{A^2(0,q)},\ K=-\frac{A(0,q)D'(0,q)}{B(0,q)C(0,q)},$$

$$k=\frac{C^2(0,r)}{A^2(0,r)},\ k'=\frac{B^2(0,r)}{A^2(0,r)},\ K'=-\frac{A(0,r)D'(0,r)}{B(0,r)C(0,r)},$$

$$\log q=-\frac{\pi K'}{K},\ \log r=-\frac{\pi K}{K'},$$

where if the identity of the two values of k or of the two values of k' were proved independently (as might doubtless be done), the required theorem (r the same function of k' that q is of k) would follow conversely: and thence the other equations of the system.

66. We have in the 'Fundamenta Nova,' p. 75, the equation

$$\frac{H(iu,k)}{\Theta(0,k)}=i\sqrt{\frac{k}{k'}}e^{\frac{\pi u^2}{4KK'}}\frac{H(u,k')}{\Theta(0,k')};$$

writing here $K'u$ instead of u the equation becomes

$$\frac{H(iK'u,k)}{\Theta(0,k)}=i\sqrt{\frac{k}{k}}\exp\left(\frac{1}{4}\frac{\pi K'}{K}u^2\right).\frac{H(K'u,k')}{\Theta(0,k)}$$

or what is the same thing

$$\frac{D\left(\frac{au}{\pi i},q\right)}{C(0,q)}=i\sqrt{\frac{k}{k'}}\exp\left(-\tfrac{1}{4}u^2\log q\right).\frac{D(u,r)}{C(0,r)}$$

which can be readily identified with the foregoing equation between $D\left(\frac{au}{\pi i},q\right)$ and $D(u,r)$. But the real meaning of the transformation is best seen by means of the double-product formulæ.

THIRD PART—THE DOUBLE THETA-FUNCTIONS.

Notations, &c.

67. We have here 16 functions $\vartheta\binom{\alpha\beta}{\gamma\delta}(u,v)$: this notation by characteristics, containing each of them four numbers, is too cumbrous for ordinary use, and I therefore replace it by the current-number notation, in which the characteristics are denoted by the series of numbers 0, 1, 2, . . . 15: we cannot in place of this introduce the single-and-double-letter notation A, B, . . . AB, &c., for there is not here any correspondence of the two notations, nor is there anything in the definition of the functions which in

anywise suggests the single-and-double-letter notation: this first presents itself in connexion with the relations between the functions given by the product-theorem and as the product theorem is based upon the notation by characteristics, it is proper to present the theorem in this notation, or in the equivalent current-number notation and then to show how by the relations thus obtained between the functions we are led to the single-and-double-letter notation.

68. There are some other notations which may be referred to: and for showing the correspondence between them I annex the following table.—

THE double theta-functions.

Asterisk denotes the odd functions.	1 Current number	2 Character	3 Single-and-double letter, CAYLEY	4 GÖPEL	5 GÖPEL CAYLEY	6 ROSENHAIN	7 WEIERSTRASS	8 KUMMER
	ϑ_0	ϑ^{00}_{00}	BD	P''	P_3	ϑ_{22}	ϑ_5	12
	1	$\begin{smallmatrix}10\\00\end{smallmatrix}$	CE	R'	R_3	32	4	8
	2	$\begin{smallmatrix}01\\00\end{smallmatrix}$	CD	Q''	Q_3	23	01	10
	3	$\begin{smallmatrix}11\\00\end{smallmatrix}$	BE	S'	S_3	33	23	6
	4	$\begin{smallmatrix}00\\10\end{smallmatrix}$	AC	P'	P_1	02	34	4
*	5	$\begin{smallmatrix}10\\10\end{smallmatrix}$	C	iR'	R_1	12	3	16
	6	$\begin{smallmatrix}01\\10\end{smallmatrix}$	AB	Q'	Q_1	03	2	2
*	7	$\begin{smallmatrix}11\\10\end{smallmatrix}$	B	iS'	S_1	13	24	14
	8	$\begin{smallmatrix}00\\01\end{smallmatrix}$	BC	P'	P_2	20	12	9
	9	$\begin{smallmatrix}10\\01\end{smallmatrix}$	DE	R'	R_2	30	03	5
*	10	$\begin{smallmatrix}01\\01\end{smallmatrix}$	F	iQ''	Q_2	21	02	11
*	11	$\begin{smallmatrix}11\\01\end{smallmatrix}$	A	iS	S_2	31	13	7
	12	$\begin{smallmatrix}00\\11\end{smallmatrix}$	AD	P	P	00	0	1
*	13	$\begin{smallmatrix}10\\11\end{smallmatrix}$	D	iR	R	10	04	13
*	14	$\begin{smallmatrix}01\\11\end{smallmatrix}$	E	iQ	Q	01	1	3
	15	$\begin{smallmatrix}11\\11\end{smallmatrix}$	AE	S	S	11	14	15

69. These are the notations :—

1. By current-numbers. It may be remarked that the series was taken 0, 1, ... 15 instead of 1, 2, ... 16, in order that 0 might correspond to the characteristic $\begin{smallmatrix}00\\00\end{smallmatrix}$;

2. By characteristics;

3. By single-and-double letters;

4. GOPEL'S, in his paper above referred to, and

5. The same as used in my paper above referred to;

6. ROSENHAIN'S, in his paper above referred to;

7. WEIERSTRASS', as quoted by KONIGSBERGER in his paper "Ueber die Transformation der *Abel*schen Functionen erster Ordnung," 'Crelle-Borchardt,' t. 64 (1865), p. 17, and by BORCHARDT in his paper above referred to;

8. Not a theta-notation, but the series of current numbers used in KUMMER'S Memoir "Ueber die algebraischen Strahlen-systeme," 'Berl. Abh.' 1866, for the nodes of his 16-nodal quartic surface, and connected with the double theta-functions in my paper above referred to.

But in the present memoir only the first three columns of the table need be attended to.

70. It will be convenient to introduce here some other remarks as to notation, &c.

The letter c is used in connexion with the zero values $u=0$, $v=0$ of the arguments, viz.:—

$$\vartheta_0,\ \vartheta_1,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4,\ \vartheta_6,\ \vartheta_8,\ \vartheta_9,\ \vartheta_{12},\ \vartheta_{15}$$

are even functions, and the corresponding zero-functions are denoted by

$$c_0,\ c_1,\ c_2,\ c_3,\ c_4,\ c_6,\ c_8,\ c_9,\ c_{12},\ c_{15};$$

there are thus 10 constants c.

When (u, v) are indefinitely small each of these functions is of course equal to its zero-value *plus* a quadric term in (u, v), and we may write in general

$$\vartheta=c+\tfrac{1}{2}(c''',\ c^{\text{iv}},\ c^{\text{v}}\between u,\ v)^2;$$

there are thus 30 constants c''', c^{iv}, c^{v}.

The remaining functions

$$\vartheta_5,\ \vartheta_7,\ \vartheta_{10},\ \vartheta_{11},\ \vartheta_{13},\ \vartheta_{14}$$

are odd functions vanishing for $u=0$, $v=0$, and when these arguments are indefinitely small we may write in general

$$\vartheta=(c',\ c''\between u,\ v)$$

there are thus 12 constants c', c''.

71. All these constants are in the first instance given as transcendental functions of the parameters, or say rather of exp. a, exp. h, exp. b (which exponentials correspond to the q of the single theory): viz., in a notation which will be readily understood, the constants c, c''', c^{iv}, c^{v} of the even functions are

$$\Sigma \exp \begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix};$$

$$-\tfrac{1}{2}\pi^2\Sigma\ (m+\alpha)^2,\ 2(m+\alpha)(n+\beta),\ (n+\beta)^2,\ \exp \begin{pmatrix} m+\alpha, & n+\delta \\ \gamma & \delta \end{pmatrix};$$

and the constants c', c'' of the odd functions are

$$\tfrac{1}{2}\pi i\Sigma\ (m+\alpha),\ (n+\beta),\ \exp \begin{pmatrix} m+\alpha, & n+\beta \\ \gamma & \delta \end{pmatrix}.$$

72. It is convenient for the verification of results and otherwise, to have the values of the functions, belonging to small values of exp $(-a)$, exp $(-b)$; if to avoid fractional exponents we regard these and exp $(-h)$ as fourth powers, and write

$$\exp(-a)=\mathrm{Q}^4,\ \exp(-h)=\mathrm{R}^4,\ \exp(-b)=\mathrm{S}^4,$$

also

$$\mathrm{QR^2S}=\Lambda,\ \mathrm{QR^{-2}S}=\Lambda',\ \text{and therefore } \Lambda\Lambda'=\mathrm{Q^2S^2},$$

then attending only to the lowest powers of Q, S we find without difficulty

$\vartheta_0(u)=\ 1,$	and therefore also	$c_0=1,$
$\vartheta_1\ =\ 2\mathrm{Q}\cos\tfrac{1}{2}\pi u,$		$c_1=2\mathrm{Q},$
$\vartheta_2\ =\ 2\mathrm{S}\cos\tfrac{1}{2}\pi v,$		$c_2=2\mathrm{S},$
$\vartheta_3\ =\ 2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v),$		$c_3=2\Lambda+2\Lambda',$
$\vartheta_4\ =\ 1,$		$c_4=1,$
$\vartheta_5\ =-2\mathrm{Q}\sin\tfrac{1}{2}\pi u,$		
$\vartheta_6\ =\ 2\mathrm{S}\cos\tfrac{1}{2}\pi v,$		$c_6=2\mathrm{S},$
$\vartheta_7\ =-2\Lambda\sin\tfrac{1}{2}\pi(u+v)-2\Lambda'\sin\tfrac{1}{2}\pi(u-v),$		
$\vartheta_8\ =\ 1,$		$c_8=1,$
$\vartheta_9\ =\ 2\mathrm{Q}\cos\tfrac{1}{2}\pi u,$		$c_9=2\mathrm{Q},$
$\vartheta_{10}=-2\mathrm{S}\sin\tfrac{1}{2}\pi v,$		
$\vartheta_{11}=-2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v),$		
$\vartheta_{12}=\ 1,$		$c_{12}=1,$
$\vartheta_{13}=-2\mathrm{Q}\sin\tfrac{1}{2}\pi u,$		
$\vartheta_{14}=-2\mathrm{S}\sin\tfrac{1}{2}\pi v,$		
$\vartheta_{15}=-2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v),$		$c_{15}=-2\Lambda+2\Lambda'.$

73. The expansions might be carried further; we have for instance

$$\begin{array}{lll}
\vartheta_0(u)=1+2Q^4\cos\pi u+2S^4\cos\pi v, & & c_0=1+2Q^4+2S^4,\\
\vartheta_4\ \ =1-2Q^4\ \ ,,\ \ +2S^4\ \ ,,\ \ , & & c_4=1-2Q^4+2S^4,\\
\vartheta_8\ \ =1+2Q^4\ \ ,,\ \ -2S^4\ \ ,,\ \ , & & c_8=1+2Q^4-2S^4,\\
\vartheta_{12}=1-2Q^4\ \ ,,\ \ -2S^4\ \ ,,\ \ , & & c_{12}=1-2Q^4-2S^4,
\end{array}$$

$$\vartheta_1 = 2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u+2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v),$$

$$c_1=2Q+2Q^9+2A+2A',$$

$$\vartheta_5 = -2Q\sin\tfrac{1}{2}\pi u+2Q^9\sin\tfrac{3}{2}\pi u-2A\sin\tfrac{1}{2}\pi(u+2v)-2A'\sin\tfrac{1}{2}\pi(u-2v)$$

$$\vartheta_9 = 2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u-2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v),$$

$$c_9=2Q+2Q^9-2A-2A',$$

$$\vartheta_{13} = -2Q\sin\tfrac{1}{2}\pi u+2Q^9\sin\tfrac{3}{2}\pi u+2A\sin\tfrac{1}{2}\pi(u+2v)+2A'\sin\tfrac{1}{2}\pi(u-2v),$$

in which last formulæ

$$A=QR^4S^4,=\frac{\Lambda^2S^2}{Q};\ A'=QR^{-4}S^4,=\frac{\Lambda^{12}S^2}{Q}.$$

74. In the single-and-double-letter notation we have six letters A, B, C, D, E, F; and the duads AB, AC, ... DE are used as abbreviations for the double triads ABF, CDE, &c., the letter F always accompanying the expressed duad; there are thus in all six single-letter symbols, and 10 double-letter symbols, in all 16 symbols, corresponding to the double-theta functions, as already mentioned in the order

ϑ 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
BD,	CE,	CD,	BE,	AC,	C,	AB,	B,	BC,	DE,	F,	A,	AD,	D,	E,	AE

where observe that the single letters C, B, F, A, D, E correspond to the odd functions 5, 7, 10, 11, 13, 14 respectively.

75. The ground of the notation is as follows:—

The algebraical relations between the double theta-functions are such that introducing six constant quantities a, b, c, d, e, f and two variable quantities (x, y) it is allowable to express the 16 functions as proportional to given functions of these quantities $(a, b, c, d, e, f;\ x, y)$; viz.. there are six functions the notations of which depend on the single letters a, b, c, d, e, f respectively, and 10 functions the notations of which depend on the pairs $ab, ac, ad, ae, bc, bd, be, cd, ce, de$ respectively: each of the 16 functions is in fact proportional to the product of two factors, viz.: a constant factor depending only on (a, b, c, d, e, f), and a variable factor containing also x and y. Attending in the first instance to the variable factors, and writing for shortness

$$a-x,\ b-x,\ c-x,\ d-x,\ e-x,\ f-x=\mathrm{a, b, c, d, e, f}\quad;\ x-y=\theta;$$
$$a-y,\ b-y,\ c-y,\ d-y,\ e-y,\ f-y=\mathrm{a}_{,},\ \mathrm{b}_{,},\ \mathrm{c}_{,},\ \mathrm{d}_{,},\ \mathrm{e}_{,},\ \mathrm{f}_{,};$$

these are of the forms

$$\sqrt{a}=\sqrt{\mathrm{aa}_{,}},\ \sqrt{ab}=\frac{1}{\theta}\{\sqrt{\mathrm{abfc}_{,}\mathrm{d}_{,}\mathrm{e}_{,}}+\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{f}_{,}\mathrm{cde}}\}$$

and I remark that to avoid ambiguity the squares of these expressions are throughout written as $(\sqrt{a})^2$ and $(\sqrt{ab})^2$ respectively.

76. There is for the constant factors a like single-and-double-letter notation which will be mentioned presently, but in the first instance I use for the even functions the before mentioned 10 letters c, and for the odd ones six letters k. I assume that the values $x, y=\infty, \infty$ (ratio not determined) correspond to the values $u=0, v=0$ of the arguments; and that ω is a function of (x, y) which, when (x, y) are interchanged, changes only its sign, and which for indefinitely large values of (x, y) becomes $=\frac{x-y}{(xy)^{\frac{1}{2}}}$. This being so, we write (adding a second column which will be afterwards explained)

$$\begin{array}{ll}
\vartheta 0=\mathrm{BD}=\omega c_0\sqrt{bd}, & c_0=\lambda\sqrt[4]{bd}, \\
1=\mathrm{CE}=\text{,,}c_1\sqrt{ce}, & c_1=\text{,,}\sqrt[4]{ce}, \\
2=\mathrm{CD}=\text{,,}c_2\sqrt{cd}, & c_2=\text{,,}\sqrt[4]{cd}, \\
3=\mathrm{BE}=\text{,,}c_3\sqrt{be}, & c_3=\text{,,}\sqrt[4]{be}, \\
4=\mathrm{AC}=\text{,,}c_4\sqrt{ac}, & c_4=\text{,,}\sqrt[4]{ac}, \\
5=\ \mathrm{C}\ =\text{,,}k_5\sqrt{c}, & k_5=\text{,,}\sqrt[4]{c}, \\
6=\mathrm{AB}=\text{,,}c_6\sqrt{ab}, & c_6=\text{,,}\sqrt[4]{ab}, \\
7=\ \mathrm{B}\ =\text{,,}k_7\sqrt{b}, & k_7=\text{,,}\sqrt[4]{b}, \\
8=\mathrm{BC}=\text{,,}c_8\sqrt{bc}, & c_8=\text{,,}\sqrt[4]{bc}, \\
9=\mathrm{DE}=\text{,,}c_9\sqrt{de}, & c_9=\text{,,}\sqrt[4]{de}, \\
10=\ \mathrm{F}\ =\text{,,}k_{10}\sqrt{f}, & k_{10}=\text{,,}\sqrt[4]{f}, \\
11=\ \mathrm{A}\ =\text{,,}k_{11}\sqrt{a}, & k_{11}=\text{,,}\sqrt[4]{a}, \\
12=\mathrm{AD}=\text{,,}c_{12}\sqrt{ad}, & c_{12}=\text{,,}\sqrt[4]{ad}, \\
13=\ \mathrm{D}\ =\text{,,}k_{13}\sqrt{d}, & k_{13}=\text{,,}\sqrt[4]{d}, \\
14=\ \mathrm{E}\ =\text{,,}k_{14}\sqrt{e}, & k_{14}=\text{,,}\sqrt[4]{e}, \\
15=\mathrm{AE}=\text{,,}c_{15}\sqrt{ae}, & c_{15}=\text{,,}\sqrt[4]{ae},
\end{array}$$

viz.: here, on writing $x, y=\infty, \infty$, each of the functions $\sqrt{bd}$, &c. becomes $=2\frac{(xy)^{\frac{1}{2}}}{x-y}$; and each of the functions $\sqrt{a}$, &c., becomes $=\sqrt{xy}$; hence by reason of the assumed form of ω, the odd functions each vanish (their evanescent values being proportional

to k_5, k_7, k_{10}, k_{11}, k_{13}, k_{14} respectively), while the even functions become equal to c_0, c_1, c_2, c_3, c_4, c_6, c_8, c_9, c_{12}, c_{15} respectively.

Observe further that on interchanging x, y, the even functions remain unaltered, while the odd functions change their sign; that is, the interchange of x, y corresponds to the change u, v into $-u$, $-v$.

77. As to the values of the 10 c's and the six k's in terms of (a, b, c, d, e, f) these are proportional to fourth roots, $\sqrt{\overline{\overline{a}}}$, &c., $\sqrt[4]{\overline{ab}}$, &c.; in $\sqrt[4]{\overline{a}}$, a is in the first instance regarded as standing for the pentad $bcdef$, and then this is used to denote a product of differences $bc.bd.be\ bf.cd.ce.cf.de.df.ef$; similarly ab is in the first instance regarded as standing for the double triad $abf.cde$, and then each of these triads is used to denote a product of differences, $ab.af.bf$ and $cd.ce.de$ respectively. The order of the letters is always the alphabetical one, viz.: the single letters and duads denote pentads and double triads, thus:

$$\begin{array}{ll} a=bcdef, & ab=abf.cde, \\ b=acdef, & ac=acf.bde, \\ c=abdef, & ad=adf.bce, \\ d=abcef, & ae=aef.bcd, \\ e=abcdf, & bc=bcf.ade, \\ f=abcde, & bd=bdf.ace, \\ & be=bef.acd, \\ & cd=cdf.abe, \\ & ce=cef.abd, \\ & de=def.abc. \end{array}$$

There is no fear of ambiguity in writing (and we accordingly write) the squares of $\sqrt[4]{a}$ and $\sqrt[4]{ab}$ as $\sqrt{\overline{\overline{a}}}$ and $\sqrt{\overline{\overline{ab}}}$ respectively; the fourth powers are written $(\sqrt{\overline{\overline{a}}})^2$ and $(\sqrt{\overline{\overline{ab}}})^2$; the double stroke of the radical symbol $\sqrt{\overline{\overline{\ }}}$ is in every case perfectly distinctive.

This being so we have as above $c_0=\lambda\sqrt[4]{\overline{\overline{bd}}}$, &c., $k_5=\lambda\sqrt[4]{\overline{a}}$, &c.: it is, however, important to notice that the fourth roots in question do not denote positive values, but they are fourth roots each taken with its proper sign ($+$, $-$, $+i$, $-i$, as the case may be) so as to satisfy the identical relations which exist between the sixteen constants; and it is not easy to determine the signs.

The x, y are connected with the u, v by the differential relations

$$\sigma du+\tau dv=-\tfrac{1}{2}\left\{\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}}\right\},$$

$$\varpi du+\rho dv=-\tfrac{1}{2}\left\{\frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}}\right\},$$

where X=abcdef, $Y=a_{,}b_{,}c_{,}d_{,}e_{,}f_{,}$; which equations contain the constants ϖ, ρ, σ, τ, the values of which will be afterwards connected with the other constants.

78. The c's are expressed as functions of four quantities α, β, γ, δ, and connected with each other, and with the constants (a, b, c, d, e, f) by the formulæ

$$\begin{array}{rll}
c^2 & & \\
\overline{0} = \alpha^2+\beta^2+\gamma^2+\delta^2 & = \omega_0^2\sqrt{bd}, \\
1 = 2(\alpha\beta+\gamma\delta) & = \;,, \;\sqrt{ce}, \\
2 = 2(\alpha\gamma+\beta\delta) & = \;,, \;\sqrt{cd}, \\
3 = 2(\alpha\delta+\beta\gamma) & = \;,, \;\sqrt{be}, \\
4 = \alpha^2-\beta^2+\gamma^2-\delta^2 & = \;,, \;\sqrt{ac}, \\
6 = 2(\alpha\gamma-\beta\delta) & = \;,, \;\sqrt{ab}, \\
8 = \alpha^2+\beta^2-\gamma^2-\delta^2 & = \;,, \;\sqrt{bc}, \\
9 = 2(\alpha\beta-\gamma\delta) & = \;,, \;\sqrt{de}, \\
12 = \alpha^2-\beta^2-\gamma^2-\delta^2 & = \;,, \;\sqrt{ad}, \\
15 = 2(\alpha\delta-\beta\gamma) & = \;,, \;\sqrt{ae}.
\end{array}$$

It hence appears that we can form an arrangement

$$\left|\begin{array}{ccc} c^2_{12}, & c^2_1, & c^2_6 \\ c^2_9, & -c^2_4, & c^2_3 \\ c^2_2, & -c^2_{15}, & -c^2_8 \end{array}\right| \div c^2_0 \quad = \left|\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{array}\right|$$

a system of coefficients in the transformation between two sets of rectangular coordinates.

We have between the squares of these coefficients of transformation 6+9 equations

$$a^2+b^2+c^2=1,$$
$$a'^2+b'^2+c'^2=1,$$
$$a''^2+b''^2+c''^2=1,$$

$$a^2+a'^2+a''^2=1,$$
$$b^2+b'^2+b''^2=1,$$
$$c^2+c'^2+c''^2=1,$$

$$b^2+c^2=a'^2+a''^2,\quad b'^2+c'^2=a''^2+a^2,\quad b''^2+c''^2=a^2+a'^2,$$
$$c^2+a^2=b'^2+b''^2,\quad c'^2+a'^2=b''^2=b^2,\quad c''^2+a''^2=b^2+b'^2,$$
$$a^2+b^2=c^2+c''^2,\quad a'^2+b'^2=c''^2=c^2,\quad a''^2+b''^2=c^2+c'^2,$$

that is

$$\left|\begin{array}{cccc}
c^4 & c^4 & c^4 & c^4 \\
12 & +1 & +6 & -0 \\
9 & +4 & +3 & -0 \\
2 & +15 & +8 & -0 \\
12 & +9 & +2 & -0 \\
1 & +4 & +15 & -0 \\
6 & +3 & +8 & -0 \\
1 & +6 & -9 & -2 \\
6 & +12 & -4 & -15 \\
12 & +1 & -3 & -8 \\
4 & +3 & -2 & -12 \\
3 & +9 & -15 & -1 \\
9 & +4 & -8 & -6 \\
15 & +8 & -12 & -9 \\
8 & +2 & -1 & -4 \\
2 & +15 & -6 & -3
\end{array}\right| = 0;$$

and between the products a system of 6+9 equations

$$
\begin{aligned}
&a'a''+b'b''+c'c''=0,\\
&a''a+b''b+c''c=0,\\
&aa'+bb'+cc'=0,\\
&bc+b'c'+b''c''=0,\\
&ca+c'a'+c''a''=0,\\
&ab+a'b'+a''b''=0,
\end{aligned}
$$

$$
\begin{array}{lll}
a,\ b,\ c & =b'c''-b''c', & c'a''-c''a', & a'b''-a''b',\\
a',\ b',\ c' & \ \ b''c-bc'', & c''a-ca'', & ab''-a''b,\\
a'',\ b'',\ c'' & \ \ bc'-b'c, & ca'-c'a, & ab'-a'b,
\end{array}
$$

that is

c^2	c^2	c^2	c^2	c^2	c^2
9	2	+ 4	15	− 3	8
2	12	−15	1	− 8	6
12	9	− 1	4	+ 6	3
1	6	− 4	3	+15	8
6	12	+ 3	9	− 8	2
12	1	− 9	4	− 2	15
−0	12	+ 4	8	+ 3	15
−0	1	+ 3	2	+ 8	9
−0	6	− 9	15	+ 2	4
−0	9	−15	6	+ 8	1
+0	4	− 8	12	− 6	2
−0	3	−12	15	− 1	2
−0	2	+ 1	3	+ 4	6
+0	15	+ 6	9	− 3	12
+0	8	−12	4	− 9	1

=0;

each of the first set of 15 giving a homogeneous linear relation between four terms c^4; and each of the second set giving a homogeneous linear relation between three terms $c^2 . c^2$, formed with the 10 constants c. Thus the first equation is $c_{12}{}^4+c_1{}^4+c_6{}^4-c_0{}^4=0$; and so for the other lines of the two diagrams.

79. I form in the two notations the following tables:—

TABLE of the 16 KUMMER hexads.

A	A	A	A	A	B	B	B	B	C	C	C	D	D	E	A
B	C	D	E	F	C	D	E	F	D	E	F	E	F	F	B
AB	AC	AD	AE	AB	BC	BD	BE	AB	CD	CE	AC	DE	AD	AE	C
CD	BD	BC	BC	AC	AD	AC	AC	BC	AB	AB	BC	AB	BD	BE	D
CE	BE	BE	BD	AD	AE	AE	AD	BD	AE	AD	CD	AC	CD	CE	E
DE	DE	CE	CD	AE	DE	CE	CD	BE	BE	BD	CE	BC	DE	DE	F

=

11	11	11	11	11	7	7	7	7	5	5	5	13	13	14	11
7	5	13	14	10	5	13	14	10	13	14	10	14	10	10	7
6	4	14	12	6	8	0	3	6	2	1	4	9	12	15	5
2	0	8	8	4	12	4	4	8	6	6	8	6	0	3	13
1	3	3	3	12	15	15	12	0	15	12	2	4	2	1	14
9	9	1	2	15	9	1	2	3	3	0	1	8	9	9	10

80. TABLE of the 120 pairs.

A.B AC.BC AD.BD AE.BE	A.C AB.BC AD.CD AE.CE	A.D AB.BD AC.CD AE.DE	A.E AB.BE AC.CE AD.DE	A.F BC.DE BD.CE BE.CD	B.C AB.AC BD.CD BE.CE	B.D AB.AD BC.CD BE.DE	B E AB.AE BC.CE BD.DE	B.F AC.DE AD.CE AE.CD	C.D AC.AD BC.BD CE.DE	C.E AC.AE BC.BE CD.DE	C.F AB.DE AD.BE AE.BD	D.E AD.AE BD.BE CD.CE	D.F AB.CE AC.BE AE.BC	E.F AB.CD AC.BD AD.BC
F.AB C.DE D.CE E.CD	F.AC B.DE D.BE E.BD	F.AD B.CE C.BE E.BC	F.AE B.CD C.BD D.BC	B.AB C.AC D.AD E.AE	F.BC A.DE D.AE E.AD	F.BD A.CE C.AE E.AC	F.BE A.CD C.AD D.AC	A.AB C BC D.BD E.BE	F.CD A.BE B.AE E.AB	F.CE A.BD B.AD D.AB	A.AC B.BC D.CD E CE	F.DE A.BC B.AC C.AB	A.AD B.BD C.CD E.DE	A.AE B.BE C.CE D.DE

=

11.7 4.8 12.0 15.3	11.5 6.8 12.2 15.1	11.13 6.0 4.2 15.9	11.14 6.3 4.1 12.9	11.10 8.9 0.1 3.2	7.5 6.4 0.2 3.1	7.13 6.12 8.2 3.9	7.14 6.15 8.1 0.9	7.10 4.9 12.1 15.2	5.13 4.12 8.0 1.9	5.14 4.15 8.3 2.9	5.10 6.9 12 3 15.0	13.14 12.15 0.3 2.1	13.10 6.1 4.3 15.8	14.10 6.2 4.0 12.8
10.6 5.9 13.1 14.2	10.4 7.9 13.3 14.0	10.12 7.1 5.3 14.8	10.15 7.2 5.0 13.8	7.6 5.4 13.12 14.15	10.8 11.9 13.15 14.12	10.0 11.1 5.15 14.4	10.3 11.2 5.12 13.4	11.6 5.8 13.0 14.3	10.2 11.3 7.15 14.6	10.3 11.0 7.12 13.6	11.4 7.8 13.2 14.1	10.9 11.8 7.4 5.6	11.12 7.0 5.2 14.9	11.15 7.3 5.1 13.9

81. Table of the 60 Gopel tetrads.

A.B.AE.BE A.B.AD.BD A.B.AC.BC	C.D.CE.DE C.E.CD.DE C.F.AB.DE	E.F.AB.CD D.F.AB.CE D.E.CD.CE	AC.BD.AD.BC AC.BE.AE.BC AD.BE.AE.BD
A.C.AE.CE A.C.AD.CD A.C.AB.BC	B.D.BE.DE B.E.BD.DE B.F.AC.DE	E.F.AC.BD D.F.AC.BE D.E.BD.BE	AB.CD.AD.BC AB.CE.AE.BC AD.CE.AE.CD
A.D.AE.DE A.D.AC.CD A.D.AB.BD	B.C.BE.CE B.E.BC.CE B.F.AD.CE	E.F.AD.BC C.F.AD.BE C.E.CD.DE	AB.CD.AC.BD AB.DE.AE.BD AC.DE.AE.CD
A.E.AD.DE A.E.AC.CE A.E.AB.BE	B.C.BD.CD B.D.BC.CD B.F.AE.CD	D.F.AE.BC C.F.AE.BD C.D.BC.BD	AB.CE.AC.BE AB.DE.AD.BE AC.DE.AD.CE
A.F.BC.DE A.F.BD.CE A.F.BE.CD	B.C.AB.AC B.D.AB.AD B.E.AB.AE	D.E.AD.AE C.E.AC.AE C.D.AC.AD	BD.CE.BE.CD BC.DE.BE.CD BC.DE.BD.CE

=

11 7 15 3 11 7 12 0 11 7 4 8	5 13 1 9 5 14 2 9 5 10 6 9	14 10 6 2 13 10 6 1 13 14 2 1	4 0 12 8 4 3 15 8 12 3 15 0
11 5 15 1 11 5 12 2 11 5 6 8	7 13 3 9 7 14 0 9 7 10 4 9	14 10 4 0 13 10 4 3 13 14 0 3	6 2 12 8 6 1 15 8 12 1 15 2
11 13 15 9 11 13 4 2 11 13 6 0	7 5 3 1 7 14 8 1 7 10 12 1	14 10 12 8 5 10 12 3 5 14 2 9	6 2 4 0 6 9 15 0 4 9 15 2
11 14 12 9 11 14 4 1 11 14 6 3	7 5 0 2 7 13 8 2 7 10 15 2	13 10 15 8 5 10 15 0 5 13 8 0	6 1 4 3 6 9 12 3 4 9 12 1
11 10 8 9 11 10 0 1 11 10 3 2	7 5 6 4 7 13 6 12 7 14 6 15	13 14 12 15 5 14 4 15 5 13 4 12	0 1 3 2 8 9 3 2 8 9 0 1

The product-theorem, and its results.

82. The product-theorem was

$$\vartheta\binom{\alpha,\beta}{\gamma,\delta}(u+u').\vartheta\binom{\alpha',\beta'}{\gamma',\delta'}(u-u')=\Sigma\Theta^{\frac{1}{2}(\alpha+\alpha')+p,\,\frac{1}{2}(\beta+\beta')+q}_{\gamma+\gamma'\quad,\quad\delta+\delta'}(2u).\Theta^{\frac{1}{2}(\alpha-\alpha')+p,\,\frac{1}{2}(\beta-\beta')+q}_{\gamma-\gamma'\quad,\quad\delta-\delta'}(2u')$$

where only one argument is exhibited, viz.: $u+u'$, $u-u'$, $2u$, $2u'$ are written in place of $(u+u', v+v')$, $(u-u', v-v')$, $(2u, 2v)$, $(2u', 2v')$ respectively. The expression on the right hand side is always a sum of four terms, corresponding to the values (0, 0), (1, 0), (0, 1), and (1, 1) of (p, q). For the development of the results it was found convenient to use the following auxiliary diagram.

UPPER half of characteristic.

α	β	α'	β'	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')+1$	$\frac{1}{2}(\beta+\beta')$	$\frac{1}{2}(\alpha-\alpha')+1$	$\frac{1}{2}(\beta-\beta')$	$\frac{1}{2}(\alpha+\alpha')$	$\frac{1}{2}(\beta+\beta)+1$	$\frac{1}{2}(\alpha-\alpha')$	$\frac{1}{2}(\beta-\beta)+1$	$\frac{1}{2}(\alpha+\alpha)+1$	$\frac{1}{2}(\beta+\beta)+1$	$\frac{1}{2}(\alpha-\alpha)+1$	$\frac{1}{2}(\beta-\beta')+1$
0	0	0	0	0	0	0	0	1	0	1	0	0	1	0	1	1	1	1	1
1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1
0	1	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$
1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$	1
1	0	1	0	1	0	0	0	0	0	1	0	1	1	0	1	0	1	1	1
0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
1	1	1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$
0	0	0	1	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{1}{2}$
1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
0	1	0	1	0	1	0	0	1	1	1	0	0	0	0	1	1	0	1	1
1	1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1
0	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	1	1	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$
0	1	1	1	$\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	1
1	1	1	1	1	1	0	0	0	1	1	0	1	0	0	1	0	0	1	1

LOWER half of characteristic.

γ	δ	γ'	δ'	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$	$\gamma+\gamma'$	$\delta+\delta'$	$\gamma-\gamma'$	$\delta-\delta'$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	1	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0
1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0	0
0	1	1	0	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1
1	1	1	0	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0	1
0	0	0	1	0	1	0	−1	0	1	0	−1	0	1	0	−1	0	1	0	−1
1	0	0	1	1	1	1	−1	1	1	1	−1	1	1	1	−1	1	1	1	−1
0	1	0	1	0	2	0	0	0	2	0	0	0	2	0	0	0	2	0	0
1	1	0	1	1	2	1	0	1	2	1	0	1	2	1	0	1	2	1	0
0	0	1	1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
1	0	1	1	2	1	0	−1	2	1	0	−1	2	1	0	−1	2	1	0	−1
0	1	1	1	1	2	−1	0	0	2	−1	0	0	2	−1	0	1	2	−1	0
1	1	1	1	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0

83. The upper characters of the Θ's have thus the values 0, 1, $\tfrac{1}{2}$, $\tfrac{3}{2}$, the lower characters are originally 2, 1, 0, or -1, and these have when necessary to be by the addition or subtraction of 2 reduced to 0 or 1; the effect of this change is either to leave the Θ unaltered, or to multiply it by -1 or $\pm i$, as follows

$$\begin{array}{l|l}
\Theta^{0}_{\gamma\pm2} = \ \ \Theta^{0}_{\gamma}, & \Theta^{\frac12}_{\gamma+2} = \ \ i\Theta^{\frac12}_{\gamma},\ \ \Theta^{\frac12}_{\gamma-2} = -i\Theta^{\frac12}_{\gamma}, \\
\Theta^{1}_{\gamma\pm2} = -\Theta^{1}_{\gamma}, & \Theta^{\frac32}_{\gamma+2} = -i\Theta^{\frac32}_{\gamma},\ \ \Theta^{\frac32}_{\gamma+2} = \ \ i\Theta^{\frac32}_{\gamma},
\end{array}$$

where only the first column of characters is shown, but the same rule applies to the second column; and where we must of course combine the multipliers corresponding to the first and second columns respectively: for instance

$$\Theta^{\frac32\ \frac12}_{\gamma+2\ \delta+2} = (-i.-i=) -\Theta^{\frac32\ \frac12}_{\gamma\ \ \delta}.$$

Thus taking the tenth line of the upper half, and the fifth line of the lower half, we have

$$\begin{array}{c|c|cccc|cccc|cccc|cccc}
10 & 01 & \frac12 & \frac12 & \frac12 & \frac32 & \frac32 & \frac12 & \frac32 & \frac32 & \frac12 & \frac32 & \frac12 & \frac12 & \frac32 & \frac32 & \frac32 & \frac12 \\
00 & 10 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0
\end{array}$$

giving the value of $\vartheta^{1\ 0}_{0\ 0}(u+u').\vartheta^{0\ 1}_{1\ 0}(u-u')$: viz. this is

$$\begin{aligned}
= \ \ & \Theta^{\frac12\ \frac12}_{1\ \ 0}(2u)\,.\,\Theta^{\ \frac12\ \frac32}_{-1\ 0}(2u') = & & i\Theta^{\frac12\ \frac12}_{1\ \ 0}(2u)\,.\,\Theta^{\frac12\ \frac32}_{1\ \ 0}(2u') \\
& +\Theta^{\frac32\ \frac12}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\ \frac32\ \frac32}_{-1\ 0}(\text{,,}) & & -i\Theta^{\frac32\ \frac12}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\frac32\ \frac32}_{1\ \ 0}(\text{,,}) \\
& +\Theta^{\frac12\ \frac32}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\ \frac12\ \frac12}_{-1\ 0}(\text{,,}) & & +i\Theta^{\frac12\ \frac32}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\frac12\ \frac12}_{1\ \ 0}(\text{,,}) \\
& +\Theta^{\frac32\ \frac32}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\ \frac12\ \frac12}_{-1\ 0}(\text{,,}) & & -i\Theta^{\frac32\ \frac32}_{1\ \ 0}(\text{,,})\,.\,\Theta^{\frac12\ \frac12}_{1\ \ 0}(\text{,,}),
\end{aligned}$$

where the first column is the value given directly by the diagram, and which is then reduced to that given by the second column.

84. But instead of the Θ's we introduce single letters (X, Y, Z, W), (E, F, G, H), (I, J, K, L), (M, N, P, Q), with the suffixes (0, 1, 2, 3), in all 64 symbols, thus

$\Theta\,(2u) =$	00	10	01	11
00				
10				
01				
11				

	X	Y	Z	W
0				
1				
2				
3				

that is $\Theta^{00}_{00}(2u)=X$, $\Theta^{10}_{00}=Y$, &c.

$\Theta^{00}_{10}(2u)=X_1\ldots$

$\vdots$

$\Theta\,(2u) =$	$\frac{1}{2}0$	$\frac{1}{2}1$	$\frac{3}{2}0$	$\frac{3}{2}1$
00				
10				
01				
11				

	E	F	G	H
0				
1				
2				
3				

that is $\Theta^{\frac{1}{2}0}_{00}(2u)=E$, &c.

$\Theta\,(2u) =$	$0\frac{1}{2}$	$1\frac{1}{2}$	$0\frac{3}{2}$	$1\frac{3}{2}$
00				
10				
01				
11				

	I	J	K	L
0				
1				
2				
3				

The functions of $(2u')$ are denoted in like manner by accented letters

$\Theta^{00}_{00}(2u')=X'$, &c.

$\Theta\,(2u) =$	$\frac{1}{2}\frac{1}{2}$	$\frac{3}{2}\frac{1}{2}$	$\frac{1}{2}\frac{3}{2}$	$\frac{3}{2}\frac{3}{2}$
00				
10				
01				
11				

	M	N	P	Q
0				
1				
2				
3				

85. To simplify the expression of the results, instead of in each case writing down the suffixes, I have indicated them by means of the column headed "Suff."

Thus

$$| 8-0 | \qquad \vartheta^{01}_{01}u+u' \,.\, \vartheta^{00}_{00}u-u' = XX' + YY' + ZZ' + WW' \qquad \overset{\text{Suff.}}{| 2 |}$$

means that the equation is to be read

$$=X_2X_2'+Y_2Y_2'+Z_2Z_2'+W_2W_2'.$$

It is hardly necessary to mention that the | 8—0 | of the left hand column shows the current numbers of the theta-functions; viz.: the left hand side of the equation is $\vartheta_8(u+u')\,.\,\vartheta_0(u-u')$.

And by a preceding remark the single arguments $u+u'$ and $u-u'$ are written in place of $(u+u',\ v+v')$ and $(u-u',\ v-v')$ respectively.

The 256 equations now are

86. First set, 64 equations.

								Suffixes.
0—0	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{00}_{00}\, u-u$	=	$X\,X'$	$+Y\,Y'$	$+Z\,Z'$	$+W\,W'$	0
4—0	$\genfrac{}{}{0pt}{}{00}{10}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$X\,X'$	$+Y\,Y'$	$+Z\,Z'$	$+W\,W'$	1
8—0	$\genfrac{}{}{0pt}{}{00}{01}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$X\,X'$	$+Y\,Y'$	$+Z\,Z'$	$+W\,W'$	2
12—0	$\genfrac{}{}{0pt}{}{00}{11}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$X\,X'$	$+Y\,Y'$	$+Z\,Z'$	$+W\,W'$	3
0—4	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{00}_{10}\, u-u'$	=	$X\,X'$	$-Y\,Y'$	$+Z\,Z'$	$-W\,W'$	1
4—4	$\genfrac{}{}{0pt}{}{00}{10}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$X\,X'$	$-Y\,Y'$	$+Z\,Z'$	$-W\,W'$	0
8—4	$\genfrac{}{}{0pt}{}{00}{01}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$X\,X'$	$-Y\,Y'$	$+Z\,Z'$	$-W\,W'$	3
12—4	$\genfrac{}{}{0pt}{}{00}{11}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$X\,X'$	$-Y\,Y'$	$+Z\,Z'$	$-W\,W'$	2
0—8	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{00}_{01}\, u-u'$	=	$X\,X'$	$+Y\,Y'$	$-Z\,Z'$	$-W\,W'$	2
4—8	$\genfrac{}{}{0pt}{}{00}{10}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$X\,X'$	$+Y\,Y'$	$-Z\,Z'$	$-W\,W'$	3
8—8	$\genfrac{}{}{0pt}{}{00}{01}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$X\,X'$	$+Y\,Y'$	$-Z\,Z'$	$-W\,W'$	0
12—8	$\genfrac{}{}{0pt}{}{00}{11}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$X\,X'$	$+Y\,Y'$	$-Z\,Z'$	$-W\,W'$	1
0—12	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{00}_{11}\, u-u'$	=	$X\,X'$	$-Y\,Y'$	$-Z\,Z'$	$+W\,W'$	3
4—12	$\genfrac{}{}{0pt}{}{00}{10}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$X\,X'$	$-Y\,Y'$	$-Z\,Z'$	$+W\,W'$	2
8—12	$\genfrac{}{}{0pt}{}{00}{01}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$X\,X'$	$-Y\,Y'$	$-Z\,Z'$	$+W\,W'$	1
12—12	$\genfrac{}{}{0pt}{}{00}{11}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$X\,X'$	$-Y\,Y'$	$-Z\,Z'$	$+W\,W'$	0

FIRST set, 64 equations (continued).

								Suffixes.
1−1	$\vartheta^{10}_{00}\,u+u'\,.$	$\vartheta^{10}_{00}\,u-u'$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	0
5−1	$^{10}_{10}$	$^{10}_{00}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	1
9−1	$^{10}_{01}$	$^{10}_{00}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	2
13−1	$^{10}_{11}$	$^{10}_{00}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	3
1−5	$\vartheta^{10}_{00}\,u+u'$	$\vartheta^{10}_{10}\,u-u'$	$=$	$\mathrm{Y\,X'}$	$-\mathrm{X\,Y'}$	$+\mathrm{W\,Z}$	$-\mathrm{Z\,W'}$	1
5−5	$^{10}_{10}$	$^{10}_{10}$	$=$	$-\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	0
9−5	$^{10}_{01}$	$^{10}_{10}$	$=$	$\mathrm{Y\,X'}$	$-\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	3
13−5	$^{10}_{11}$	$^{10}_{10}$	$=$	$-\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	2
1−9	$\vartheta^{10}_{00}\,u+u'$	$\vartheta^{10}_{01}\,u-u'$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	2
5−9	$^{10}_{10}$	$^{10}_{01}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	3
9−9	$^{10}_{01}$	$^{10}_{01}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	0
13−9	$^{10}_{11}$	$^{10}_{01}$	$=$	$\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	1
1−13	$\vartheta^{10}_{00}\,u+u'$	$\vartheta^{10}_{11}\,u-u'$	$=$	$\mathrm{Y\,X'}$	$-\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	3
5−13	$^{10}_{10}$	$^{10}_{11}$	$=$	$-\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	2
9−13	$^{10}_{01}$	$^{10}_{11}$	$=$	$\mathrm{Y\,X'}$	$-\mathrm{X\,Y'}$	$-\mathrm{W\,Z'}$	$+\mathrm{Z\,W'}$	1
13−13	$^{10}_{11}$	$^{10}_{11}$	$=$	$-\mathrm{Y\,X'}$	$+\mathrm{X\,Y'}$	$+\mathrm{W\,Z'}$	$-\mathrm{Z\,W'}$	0

First set, 64 equations (continued).

				Suffixes.
2—2	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}\,u+u'\,.$	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}\,u-u'$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	0
6—2	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	1
10—2	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	2
14—2	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	3
2—6	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}\,u+u'\,.$	$\vartheta\begin{smallmatrix}01\\10\end{smallmatrix}\,u-u'$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	1
6—6	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	0
10—6	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	3
14—6	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	2
2—10	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}\,u+u\,.$	$\vartheta\begin{smallmatrix}01\\01\end{smallmatrix}\,u-u'$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}-\mathrm{X\,Z'}-\mathrm{Y\,W'}$	2
6—10	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=\mathrm{Z\,X'}+\mathrm{W\,Y'}-\mathrm{X\,Z'}-\mathrm{Y\,W'}$	3
10—10	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=-\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	0
14—10	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$=-\mathrm{Z\,X'}-\mathrm{W\,Y'}+\mathrm{X\,Z'}+\mathrm{Y\,W'}$	1
2—14	$\vartheta\begin{smallmatrix}01\\00\end{smallmatrix}\,u+u'\,.$	$\vartheta\begin{smallmatrix}01\\11\end{smallmatrix}\,u-u'$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}-\mathrm{X\,Z'}+\mathrm{Y\,W'}$	3
6—14	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=\mathrm{Z\,X'}-\mathrm{W\,Y'}-\mathrm{X\,Z'}+\mathrm{Y\,W'}$	2
10—14	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=-\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	1
14—14	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$=-\mathrm{Z\,X'}+\mathrm{W\,Y'}+\mathrm{X\,Z'}-\mathrm{Y\,W'}$	0

First set, 64 equations (concluded).

				Suffixes.
3–3	$\vartheta^{11}_{00}\, u+u'$.	$\vartheta^{11}_{00}\, u-u'$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	0
7–3	$^{11}_{10}$	$^{11}_{00}$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	1
11–3	$^{11}_{01}$	$^{11}_{00}$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	2
15–3	$^{11}_{11}$	$^{11}_{00}$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	3
3–7	$\vartheta^{11}_{00}\, u+u'$.	$\vartheta^{11}_{10}\, u-u'$	$=\ \mathrm{W\,X'} - \mathrm{Z\,Y'} + \mathrm{Y\,Z'} - \mathrm{X\,W'}$	1
7–7	$^{11}_{10}$	$^{11}_{10}$	$= - \mathrm{W\,X'} + \mathrm{Z\,Y'} - \mathrm{Y\,Z'} + \mathrm{X\,W'}$	0
11–7	$^{11}_{01}$	$^{11}_{10}$	$=\ \mathrm{W\,X'} - \mathrm{Z\,Y'} + \mathrm{Y\,Z'} - \mathrm{X\,W'}$	3
15–7	$^{11}_{11}$	$^{11}_{10}$	$= - \mathrm{W\,X'} + \mathrm{Z\,Y'} - \mathrm{Y\,Z'} + \mathrm{X\,W'}$	2
3–11	$\vartheta^{11}_{00}\, u+u'$.	$\vartheta^{11}_{01}\, u-u'$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} - \mathrm{Y\,Z'} - \mathrm{X\,W'}$	2
7–11	$^{11}_{10}$	$^{11}_{01}$	$=\ \mathrm{W\,X'} + \mathrm{Z\,Y'} - \mathrm{Y\,Z'} - \mathrm{X\,W'}$	3
11–11	$^{11}_{01}$	$^{11}_{01}$	$= - \mathrm{W\,X'} - \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	0
15–11	$^{11}_{11}$	$^{11}_{01}$	$= - \mathrm{W\,X'} - \mathrm{Z\,Y'} + \mathrm{Y\,Z'} + \mathrm{X\,W'}$	1
3–15	$\vartheta^{11}_{00}\, u+u'$.	$\vartheta^{11}_{11}\, u-u'$	$=\ \mathrm{W\,X'} - \mathrm{Z\,Y'} - \mathrm{Y\,Z'} + \mathrm{X\,W'}$	3
7–15	$^{11}_{10}$	$^{11}_{11}$	$= - \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} - \mathrm{X\,W'}$	2
11–15	$^{11}_{01}$	$^{11}_{11}$	$= - \mathrm{W\,X'} + \mathrm{Z\,Y'} + \mathrm{Y\,Z'} - \mathrm{X\,W'}$	1
15–15	$^{11}_{11}$	$^{11}_{11}$	$=\ \mathrm{W\,X'} - \mathrm{Z\,Y'} - \mathrm{Y\,Z'} + \mathrm{X\,W'}$	0

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87. Second set, 64 equations.

					Suffixes.
1–0	$\vartheta^{10}_{00}\, u+u'\,.\,\vartheta^{00}_{00}\, u-u'$		$=$	$EE' + GG' + FF' + HH'$	0
5–0	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$=$	$EE' + GG' + FF' + HH'$	1
9–0	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$=$	$EE' + GG' + FF' + HH'$	2
13–0	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}00\\00\end{matrix}$	$=$	$EE' + GG' + FF' + HH'$	3
1–4	$\vartheta^{10}_{00}\, u+u'\,.\,\vartheta^{00}_{10}\, u-u'$		$=$	$-iEE' + iGG' - iFF' + iHH'$	1
5–4	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$=$	$iEE' - iGG' + iFF' - iHH'$	0
9–4	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$=$	$-iEE' + iGG' - iFF' + iHH'$	3
13–4	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}00\\10\end{matrix}$	$=$	$iEE' - iGG' + iFF' - iHH'$	2
1–8	$\vartheta^{10}_{00}\, u+u'\,.\,\vartheta^{00}_{01}\, u-u'$		$=$	$EE' + GG' - FF' - HH'$	2
5–8	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$=$	$EE' + GG' - FF' - HH'$	3
9–8	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$=$	$EE' + GG' - FF' - HH'$	0
13–8	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}00\\01\end{matrix}$	$=$	$EE' + GG' - FF' - HH'$	1
1–12	$\vartheta^{10}_{00}\, u+u'\,.\,\vartheta^{00}_{11}\, u-u'$		$=$	$-iEE' + iGG' + iFF' - iHH'$	3
5–12	$\begin{matrix}10\\10\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$=$	$iEE' - iGG' - iFF' + iHH'$	2
9–12	$\begin{matrix}10\\01\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$=$	$-iEE' + iGG' + iFF' - iHH'$	1
13–12	$\begin{matrix}10\\11\end{matrix}$	$\begin{matrix}00\\11\end{matrix}$	$=$	$iEE' - iGG' - iFF' + iHH'$	0

SECOND set, 64 equations (continued).

								Suffixes
0−1	$\vartheta^{00}_{00}\, u+u' .$	$\vartheta^{10}_{00}\, u-u$	=	E G′	+ G E′	+ F H′	+ H F′	0
4−1	$\begin{matrix}00\\10\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	=	E G′	+ G E′	+ F H′	+ H F′	1
8−1	$\begin{matrix}00\\01\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	=	E G′	+ G E′	+ F H′	+ H F′	2
12−1	$\begin{matrix}00\\11\end{matrix}$	$\begin{matrix}10\\00\end{matrix}$	=	E G′	+ G E′	+ F H′	+ H F′	3
0−5	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{10}_{10}\, u-u'$	=	ιE G′	−ιG E′	+ιF H′	−ιH F′	1
4−5	$\begin{matrix}00\\10\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	=	ιE G′	−ιG E′	+ιF H′	−ιH F′	0
8−5	$\begin{matrix}00\\01\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	=	ιE G	−ιG E′	+ιF H′	−ιH F′	3
12−5	$\begin{matrix}00\\11\end{matrix}$	$\begin{matrix}10\\10\end{matrix}$	=	ιE G′	−ιG E′	+ιF H′	−ιH F′	2
0−9	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{10}_{01}\, u-u'$	=	E G′	+ G E′	− F H′	− H F′	2
4−9	$\begin{matrix}00\\10\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	=	E G′	+ G E′	− F H′	− H F′	3
8−9	$\begin{matrix}00\\01\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	=	E G	+ G E′	− F H′	− H F′	0
12−9	$\begin{matrix}00\\11\end{matrix}$	$\begin{matrix}10\\01\end{matrix}$	=	E G′	+ G E′	− F H′	− H F′	1
0−13	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{10}_{11}\, u-u'$	=	ιE G′	−ιG E′	−ιF H′	+ιH F′	3
4−13	$\begin{matrix}00\\10\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	=	ιE G′	−ιG E′	−ιF H′	+ιH F′	2
8−13	$\begin{matrix}00\\01\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	=	ιE G′	−ιG E′	−ιF H′	+ιH F′	1
12−13	$\begin{matrix}00\\11\end{matrix}$	$\begin{matrix}10\\11\end{matrix}$	=	ιE G′	−ιG E′	−ιF H′	+ιH F′	0

Second set, 64 equations (continued).

				Suffixes.
3−2	$\vartheta^{11}_{00}\,u+u'$	$\vartheta^{01}_{00}\,u-u'$	$= FE' + HG' + EF' + GH'$	0
7−2	${}^{11}_{10}$	${}^{01}_{00}$	$= FE' + HG' + EF' + GH'$	1
11−2	${}^{11}_{01}$	${}^{01}_{00}$	$= FE' + HG' + EF' + GH'$	2
15−2	${}^{11}_{11}$	${}^{01}_{00}$	$= FE' + HG' + EF' + GH'$	3
3−6	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{10}\,u-u'$	$=-iFE' + iHG' - iEF' + iGH'$	1
7−6	${}^{11}_{10}$	${}^{01}_{10}$	$= iFE' - iHG' + iEF' - iGH'$	0
11−6	${}^{11}_{01}$	${}^{01}_{10}$	$=-iFE' + iHG' - iEF' + iGH'$	3
15−6	${}^{11}_{11}$	${}^{01}_{10}$	$= iFE' - iHG' + iEF' - iGH'$	2
3−10	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{01}\,u-u'$	$= FE' + HG' - EF' - GH'$	2
7−10	${}^{11}_{10}$	${}^{01}_{01}$	$= FE' + HG' - EF' - GH'$	3
11−10	${}^{11}_{01}$	${}^{01}_{01}$	$=- FE' - HG' + EF' + GH'$	0
15−10	${}^{11}_{11}$	${}^{01}_{01}$	$=- FE' - HG' + EF' + GH'$	1
3−14	$\vartheta^{11}_{00}\,u+u'\,.$	$\vartheta^{01}_{11}\,u-u'$	$=-iFE' + iHG' + iEF' - iGH'$	3
7−14	${}^{11}_{10}$	${}^{01}_{11}$	$= iFE' - iHG' - iEF' + iGH'$	2
11−14	${}^{11}_{01}$	${}^{01}_{11}$	$= iFE' - iHG' - iEF' + iGH'$	1
15−14	${}^{11}_{11}$	${}^{01}_{11}$	$=-iFE' + iHG' + iEF' - iGH'$	0

SECOND set, 64 equations (concluded).

			Suffixes
2—3	$\vartheta^{01}_{00}\,u+u'\,.\,\vartheta^{11}_{00}\,u-u'$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	0
6—3	$\begin{matrix}01\\10\end{matrix}\qquad\begin{matrix}11\\00\end{matrix}$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	1
10—3	$\begin{matrix}01\\01\end{matrix}\qquad\begin{matrix}11\\00\end{matrix}$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	2
14—3	$\begin{matrix}01\\11\end{matrix}\qquad\begin{matrix}11\\00\end{matrix}$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	3
2—7	$\vartheta^{01}_{00}\,u+u'\,.\,\vartheta^{11}_{10}\,u-u'$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	1
6—7	$\begin{matrix}01\\10\end{matrix}\qquad\begin{matrix}11\\10\end{matrix}$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	0
10—7	$\begin{matrix}01\\01\end{matrix}\qquad\begin{matrix}11\\10\end{matrix}$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	3
14—7	$\begin{matrix}01\\11\end{matrix}\qquad\begin{matrix}11\\10\end{matrix}$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	2
2—11	$\vartheta^{01}_{00}\,u+u'\,.\,\vartheta^{11}_{01}\,u-u'$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} - \mathrm{E\,H'} - \mathrm{G\,F'}$	2
6—11	$\begin{matrix}01\\10\end{matrix}\qquad\begin{matrix}11\\01\end{matrix}$	$=\ \mathrm{F\,G'} + \mathrm{H\,E'} - \mathrm{E\,H'} - \mathrm{G\,F'}$	3
10—11	$\begin{matrix}01\\01\end{matrix}\qquad\begin{matrix}11\\01\end{matrix}$	$=-\ \mathrm{F\,G'} - \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	0
14—11	$\begin{matrix}01\\11\end{matrix}\qquad\begin{matrix}11\\01\end{matrix}$	$=-\ \mathrm{F\,G'} - \mathrm{H\,E'} + \mathrm{E\,H'} + \mathrm{G\,F'}$	1
2—15	$\vartheta^{01}_{00}\,u+u'\,.\,\vartheta^{11}_{11}\,u-u'$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} - i\mathrm{E\,H'} + i\mathrm{G\,F'}$	3
6—15	$\begin{matrix}01\\10\end{matrix}\qquad\begin{matrix}11\\11\end{matrix}$	$=\ i\mathrm{F\,G'} - i\mathrm{H\,E'} - i\mathrm{E\,H'} + i\mathrm{G\,F'}$	2
10—15	$\begin{matrix}01\\01\end{matrix}\qquad\begin{matrix}11\\11\end{matrix}$	$=-i\mathrm{F\,G'} + i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	1
14—15	$\begin{matrix}01\\11\end{matrix}\qquad\begin{matrix}11\\11\end{matrix}$	$=-i\mathrm{F\,G'} + i\mathrm{H\,E'} + i\mathrm{E\,H'} - i\mathrm{G\,F'}$	0

88. Third set, 64 equations.

				Suffixes.
2−0	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{00}_{00}\,u-u'$	$= \mathrm{I\,I'} + \mathrm{J\,J'} + \mathrm{K\,K'} + \mathrm{L\,L'}$	0
6−0	$^{01}_{10}$	$^{00}_{00}$	$= \mathrm{I\,I'} + \mathrm{J\,J'} + \mathrm{K\,K'} + \mathrm{L\,L'}$	1
10−0	$^{01}_{01}$	$^{00}_{00}$	$= \mathrm{I\,I'} + \mathrm{J\,J'} + \mathrm{K\,K'} + \mathrm{L\,L'}$	2
14−0	$^{01}_{11}$	$^{00}_{00}$	$= \mathrm{I\,I'} + \mathrm{J\,J'} + \mathrm{K\,K'} + \mathrm{L\,L'}$	3
2−4	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{00}_{10}\,u-u'$	$= \mathrm{I\,I'} - \mathrm{J\,J'} + \mathrm{K\,K'} - \mathrm{L\,L'}$	1
6−4	$^{01}_{10}$	$^{00}_{10}$	$= \mathrm{I\,I'} - \mathrm{J\,J'} + \mathrm{K\,K'} - \mathrm{L\,L'}$	0
10−4	$^{01}_{01}$	$^{00}_{10}$	$= \mathrm{I\,I'} - \mathrm{J\,J'} + \mathrm{K\,K'} - \mathrm{L\,L'}$	3
14−4	$^{01}_{11}$	$^{00}_{10}$	$= \mathrm{I\,I'} - \mathrm{J\,J'} + \mathrm{K\,K'} - \mathrm{L\,L'}$	2
2−8	$\vartheta^{01}_{00}\,u+u'$	$\vartheta^{00}_{01}\,u-u'$	$= -i\mathrm{I\,I'} - i\mathrm{J\,J'} + i\mathrm{K\,K'} + i\mathrm{L\,L'}$	2
6−8	$^{01}_{10}$	$^{00}_{01}$	$= -i\mathrm{I\,I'} - i\mathrm{J\,J'} + i\mathrm{K\,K'} + i\mathrm{L\,L'}$	3
10−8	$^{01}_{01}$	$^{00}_{01}$	$= i\mathrm{I\,I'} + i\mathrm{J\,J'} - i\mathrm{K\,K'} - i\mathrm{L\,L'}$	0
14−8	$^{01}_{11}$	$^{00}_{01}$	$= i\mathrm{I\,I'} + i\mathrm{J\,J'} - i\mathrm{K\,K'} - i\mathrm{L\,L'}$	1
2−12	$\vartheta^{01}_{00}\,u+u'$.	$\vartheta^{00}_{11}\,u-u'$	$= -i\mathrm{I\,I'} + i\mathrm{J\,J'} + i\mathrm{K\,K'} - i\mathrm{L\,L'}$	3
6−12	$^{01}_{10}$	$^{00}_{11}$	$= -i\mathrm{I\,I'} + i\mathrm{J\,J'} + i\mathrm{K\,K'} - i\mathrm{L\,L'}$	2
10−12	$^{01}_{01}$	$^{00}_{11}$	$= i\mathrm{I\,I'} - i\mathrm{J\,J'} - i\mathrm{K\,K'} + i\mathrm{L\,L'}$	1
14−12	$^{01}_{11}$	$^{00}_{11}$	$= i\mathrm{I\,I'} - i\mathrm{J\,J'} - i\mathrm{K\,K'} + i\mathrm{L\,L'}$	0

THIRD set, 64 equations (continued)

								Suffixes
3−1	$\vartheta^{11}_{00}\,u+u$	$\vartheta^{10}_{00}\,u-u$	$=$	$J\,I$	$+\,I\,J$	$+\,L\,K$	$+\,K\,L$	0
7−1	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$J\,I$	$+\,I\,J$	$+\,L\,K$	$+\,K\,L$	1
11−1	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$J\,I$	$+\,I\,J$	$+\,L\,K$	$+\,K\,L$	2
15−1	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$J\,I$	$+\,I\,J$	$+\,L\,K$	$+\,K\,L$	3
3−5	$\vartheta^{11}_{00}\,u+u$	$\vartheta^{10}_{10}\,u-u$	$=$	$J\,I$	$-\,I\,J$	$+\,L\,K$	$-\,K\,L$	1
7−5	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=-$	$J\,I$	$+\,I\,J$	$-\,L\,K$	$+\,K\,L$	0
11−5	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$J\,I$	$-\,I\,J$	$+\,L\,K$	$-\,K\,L$	3
15−5	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=-$	$J\,I$	$+\,I\,J$	$-\,L\,K$	$+\,K\,L$	2
3−9	$\vartheta^{11}_{00}\,u+u$	$\vartheta^{10}_{01}\,u-u$	$=-$	$\iota J\,I$	$-\iota I\,J$	$+\,L\,K$	$+\iota K\,L$	2
7−9	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=-$	$\iota J\,I$	$-\iota I\,J$	$+\iota L\,K$	$+\iota K\,L$	3
11−9	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$\iota J\,I$	$+\iota I\,J$	$-\iota L\,K$	$-\iota K\,L$	0
15−9	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$\iota J\,I$	$+\iota I\,J$	$-\iota L\,K$	$-\iota K\,L$	1
3−13	$\vartheta^{11}_{00}\,u+u$	$\vartheta^{10}_{11}\,u-u$	$=-$	$\iota J\,I$	$+\iota I\,J$	$+\iota L\,K$	$-\iota K\,L$	3
7−13	$\begin{smallmatrix}11\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$\iota J\,I$	$-\iota I\,J$	$-\iota L\,K$	$+\iota K\,L$	2
11−13	$\begin{smallmatrix}11\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$\iota J\,I$	$-\iota I\,J$	$-\iota L\,K$	$+\iota K\,L$	1
15−13	$\begin{smallmatrix}11\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=-$	$\iota J\,I$	$+\iota I\,J$	$+\iota L\,K$	$-\iota K\,L$	0

THIRD set, 64 equations (continued).

					Suffixes.
0−2	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{01}_{00}\, u-u'$	=	$IK' + JL' + KI' + LJ'$	0
4−2	$^{00}_{10}$	$^{01}_{00}$	=	$IK' + JL' + KI' + LJ'$	1
8−2	$^{00}_{01}$	$^{01}_{00}$	=	$IK' + JL' + KI' + LJ'$	2
12−2	$^{00}_{11}$	$^{01}_{00}$	=	$IK' + JL' + KI' + LJ'$	3
0−6	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{01}_{10}\, u-u'$	=	$IK' - JL' + KI' - LJ'$	1
4−6	$^{00}_{10}$	$^{01}_{10}$	=	$IK' - JL' + KI' - LJ'$	0
8−6	$^{00}_{01}$	$^{01}_{10}$	=	$IK' - JL' + KI' - LJ'$	3
12−6	$^{00}_{11}$	$^{01}_{10}$	=	$IK' - JL' + KI' - LJ'$	2
0−10	$\vartheta^{00}_{00}\, u+u'$	$\vartheta^{01}_{01}\, u-u'$	=	$iIK' + iJL' - iKI' - iLJ'$	2
4−10	$^{00}_{10}$	$^{01}_{01}$	=	$iIK' + iJL' - iKI' - iLJ'$	3
8−10	$^{00}_{01}$	$^{01}_{01}$	=	$iIK' + iJL' - iKI' - iLJ'$	0
12−10	$^{00}_{11}$	$^{01}_{01}$	=	$iIK' + iJL' - iKI' - iLJ'$	1
0−14	$\vartheta^{00}_{00}\, u+u'\,.$	$\vartheta^{01}_{11}\, u-u'$	=	$iIK' - iJL' - iKI' + iLJ'$	3
4−14	$^{00}_{10}$	$^{01}_{11}$	=	$iIK' - iJL' - iKI' + iLJ'$	2
8−14	$^{00}_{01}$	$^{01}_{11}$	=	$iIK' - iJL' - iKI' + iLJ'$	0
12−14	$^{00}_{11}$	$^{01}_{11}$	=	$iIK' - iJL' - iKI' + iLJ'$	1

Third set, 64 equations (concluded).

								Suffixes
1−3	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{11}_{00}\,u-u'$	$=$	JK'	$+IL'$	$+LI'$	$+KJ'$	0
5−3	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{11}{00}$	$=$	JK'	$+IL'$	$+LI'$	$+KJ'$	1
9−3	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{11}{00}$	$=$	JK'	$+IL'$	$+LI'$	$+KJ'$	2
13−3	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{11}{00}$	$=$	JK'	$+IL'$	$+LI'$	$+KJ'$	3
1−7	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{11}_{10}\,u-u'$	$=$	JK'	$-IL'$	$+LI'$	$-KJ'$	1
5−7	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{11}{10}$	$=-$	JK'	$+IL'$	$-LI'$	$+KJ$	0
9−7	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{11}{10}$	$=$	JK'	$-IL'$	$+LI'$	$-KJ'$	3
13−7	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{11}{10}$	$=-$	JK'	$+IL'$	$-LI'$	$+KJ'$	2
1−11	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{11}_{01}\,u-u'$	$=$	$\iota JK'$	$+\iota IL'$	$-\iota LI'$	$-\iota KJ'$	2
5−11	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{11}{01}$	$=$	$\iota JK'$	$+\iota IL'$	$-\iota LI'$	$-\iota KJ'$	3
9−11	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{11}{01}$	$=$	$\iota JK'$	$+\iota IL'$	$-\iota LI'$	$-\iota KJ'$	0
13−11	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{11}{01}$	$=$	$\iota JK'$	$+\iota IL'$	$-\iota LI'$	$-\iota KJ'$	1
1−15	$\vartheta^{10}_{00}\,u+u'$.	$\vartheta^{11}_{11}\,u-u'$	$=$	$\iota JK'$	$-\iota IL'$	$-iLI'$	$+\iota KJ'$	3
5−15	$\genfrac{}{}{0pt}{}{10}{10}$	$\genfrac{}{}{0pt}{}{11}{11}$	$=-$	$\iota JK'$	$+\iota IL'$	$+\iota LI'$	$-\iota KJ'$	2
9−15	$\genfrac{}{}{0pt}{}{10}{01}$	$\genfrac{}{}{0pt}{}{11}{11}$	$=$	$\iota JK'$	$-\iota IL'$	$-iLI'$	$+\iota KJ'$	1
13−15	$\genfrac{}{}{0pt}{}{10}{11}$	$\genfrac{}{}{0pt}{}{11}{11}$	$=-$	iJK'	$+\iota IL'$	$+\iota LI'$	$-\iota KJ'$	0

89. Fourth set, 64 equations.

					Suffixes
3−0	$\vartheta^{11}_{00}\,u+u$.	$\vartheta^{00}_{00}\,u-u'$	=	$MM'+NN'+PP'+QQ'$	0
7−0	$\genfrac{}{}{0pt}{}{11}{10}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$MM+NN'+PP'+QQ'$	1
11−0	$\genfrac{}{}{0pt}{}{11}{01}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$MM'+NN'+PP'+QQ'$	2
15−0	$\genfrac{}{}{0pt}{}{11}{11}$	$\genfrac{}{}{0pt}{}{00}{00}$	=	$MM'+NN'+PP'+QQ'$	3
3−4	$\vartheta^{11}_{00}\,u+u'$	$\vartheta^{00}_{10}\,u-u'$	=	$-iMM'+iNN'-iPP'+iQQ'$	1
7−4	$\genfrac{}{}{0pt}{}{11}{10}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$+iMM'-iNN'+iPP-iQQ'$	0
11−4	$\genfrac{}{}{0pt}{}{11}{01}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$-iMM+iNN'-iPP+iQQ$	3
15−4	$\genfrac{}{}{0pt}{}{11}{11}$	$\genfrac{}{}{0pt}{}{00}{10}$	=	$+iMM'-iNN'+iPP'-iQQ'$	2
3−8	$\vartheta^{11}_{00}\,u+u$	$\vartheta^{00}_{01}\,u-u$	=	$-iMM'-iNN'+iPP'+iQQ'$	2
7−8	$\genfrac{}{}{0pt}{}{11}{10}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$-iMM'-iNN'+iPP'+iQQ$	3
11−8	$\genfrac{}{}{0pt}{}{11}{01}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$iMM+iNN-iPP-iQQ$	0
15−8	$\genfrac{}{}{0pt}{}{11}{11}$	$\genfrac{}{}{0pt}{}{00}{01}$	=	$iMM+iNN-iPP-iQQ$	1
3−12	$\vartheta^{11}_{00}\,u+u'$	$\vartheta^{00}_{11}\,u-u$	=	$-MM'+NN'+PP'-QQ'$	3
7−12	$\genfrac{}{}{0pt}{}{11}{10}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$+MM'-NN-PP'+QQ$	2
11−12	$\genfrac{}{}{0pt}{}{11}{01}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$+MM'-NN'-PP'+QQ$	1
15−12	$\genfrac{}{}{0pt}{}{11}{11}$	$\genfrac{}{}{0pt}{}{00}{11}$	=	$-MM'+NN'+PP'-QQ'$	0

FOURTH set, 64 equations (continued).

					Suffixes.
2−1	$\vartheta^{01}_{00}\, u+u'$	$\vartheta^{10}_{00}\, u-u'$	$=$	$M N' + N M' + P Q' + Q P'$	0
6−1	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$M N' + N M' + P Q' + Q P'$	1
10−1	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$M N' + N M' + P Q' + Q P'$	2
14−1	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\00\end{smallmatrix}$	$=$	$M N' + N M' + P Q' + Q P'$	3
2−5	$\vartheta^{01}_{00}\, u+u' \,.$	$\vartheta^{10}_{10}\, u-u'$	$=$	$iM N' - iN M' + iP Q' - iQ P'$	1
6−5	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iM N' - iN M' + iP Q' - iQ P'$	0
10−5	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iM N' - iN M' + iP Q - iQ P'$	3
14−5	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$=$	$iM N' - iN M' + iP Q' - iQ P'$	2
2−9	$\vartheta^{01}_{00}\, u+u' \,.$	$\vartheta^{10}_{01}\, u-u'$	$=$	$-iM N' - iN M' + iP Q' + iQ P'$	2
6−9	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$-iM N' - iN M' + iP Q' + iQ P'$	3
10−9	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$iM N' + iN M' - iP Q' - iQ P'$	0
14−9	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$=$	$iM N' + iN M' - iP Q' - iQ P'$	1
2−13	$\vartheta^{01}_{00}\, u+u' \,.$	$\vartheta^{10}_{11}\, u-u'$	$=$	$M N' - N M' - P Q' + Q P'$	3
6−13	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$M N' - N M' - P Q' + Q P'$	2
10−13	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$- M N' + N M' + P Q' - Q P'$	1
14−13	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$=$	$- M N' + N M' + P Q' - Q P'$	0

FOURTH set, 64 equations (continued).

				Suffixes.
1−2	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{01}_{00}\, u-u'$	$= \mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} + \mathrm{Q\,N'}$	0
5−2	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= \mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} + \mathrm{Q\,N'}$	1
9−2	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= \mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} + \mathrm{Q\,N'}$	2
13−2	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\00\end{smallmatrix}$	$= \mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} + \mathrm{Q\,N'}$	3
1−6	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{01}_{10}\, u-u'$	$= -i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} + i\mathrm{Q\,N'}$	1
5−6	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= i\mathrm{M\,P'} - i\mathrm{N\,Q'} + i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	0
9−6	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= -i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} + i\mathrm{Q\,N'}$	3
13−6	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\10\end{smallmatrix}$	$= i\mathrm{M\,P'} - i\mathrm{N\,Q'} + i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	2
1−10	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{01}_{01}\, u-u'$	$= i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	2
5−10	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	3
9−10	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	0
13−10	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\01\end{smallmatrix}$	$= i\mathrm{M\,P'} + i\mathrm{N\,Q'} - i\mathrm{P\,M'} - i\mathrm{Q\,N'}$	1
1−14	$\vartheta^{10}_{00}\, u+u'$.	$\vartheta^{01}_{11}\, u-u'$	$= \mathrm{M\,P'} - \mathrm{N\,Q'} - \mathrm{P\,M'} + \mathrm{Q\,N'}$	3
5−14	$\begin{smallmatrix}10\\10\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -\mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} - \mathrm{Q\,N'}$	2
9−14	$\begin{smallmatrix}10\\01\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= \mathrm{M\,P'} - \mathrm{N\,Q'} - \mathrm{P\,M'} + \mathrm{Q\,N'}$	1
13−14	$\begin{smallmatrix}10\\11\end{smallmatrix}$	$\begin{smallmatrix}01\\11\end{smallmatrix}$	$= -\mathrm{M\,P'} + \mathrm{N\,Q'} + \mathrm{P\,M'} - \mathrm{Q\,N'}$	0

Fourth set, 64 equations (concluded).

				Suffixes
0—3	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{11}_{00}\, u-u'$	$= MQ' + NP' + PN' + QM'$	0
4—3	$^{00}_{10}$	$^{11}_{00}$	$= MQ' + NP' + PN' + QM'$	1
8—3	$^{00}_{01}$	$^{11}_{00}$	$= MQ' + NP' + PN' + QM'$	2
12—3	$^{00}_{11}$	$^{11}_{00}$	$= MQ' + NP' + PN' + QM'$	3
0—7	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{11}_{10}\, u-u'$	$= iMQ' - iNP' + iPN' - iQM'$	1
4—7	$^{00}_{10}$	$^{11}_{10}$	$= iMQ' - iNP' + iPN' - iQM'$	0
8—7	$^{00}_{01}$	$^{11}_{10}$	$= iMQ' - iNP' + iPN' - iQM'$	3
12—7	$^{00}_{11}$	$^{11}_{10}$	$= iMQ' - iNP' + iPN' - iQM'$	2
0—11	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{11}_{01}\, u-u'$	$= iMQ' + iNP' - iPN' - iQM'$	2
4—11	$^{00}_{10}$	$^{11}_{01}$	$= iMQ' + iNP' - iPN' - iQM$	3
8—11	$^{00}_{01}$	$^{11}_{01}$	$= iMQ' + iNP' - iPN' - iQM'$	0
12—11	$^{00}_{11}$	$^{11}_{01}$	$= iMQ' + iNP' - iPN' - iQM'$	1
0—15	$\vartheta^{00}_{00}\, u+u'$.	$\vartheta^{11}_{11}\, u-u'$	$= - MQ' + NP' + PN' - QM$	3
4—15	$^{00}_{10}$	$^{11}_{11}$	$= - MQ' + NP' + PN' - QM'$	2
8—15	$^{00}_{01}$	$^{11}_{11}$	$= - MQ' + NP' + PN' - QM'$	1
12—15	$^{00}_{11}$	$^{11}_{11}$	$= - MQ' + NP + PN' - QM'$	0

90. I re-arrange these in sets of 16 equations, the equations of the first or square-set of 16 being taken as they stand, but those of the other sets being combined in pairs by addition and subtraction as will be seen. And I now drop altogether the characteristics, retaining only the current numbers: thus in the set of equations next written down, the first equation is

$$\vartheta_0(u+u').\vartheta_0(u-u')=XX'+YY'+ZZ'+WW':$$

in the second set, the first equation is

$$\tfrac{1}{2}\{\vartheta_4(u+u').\vartheta_0(u-u')+\vartheta_0(u+u')\vartheta_4(u-u')\}=X_1X_1'+Z_1Z_1',$$

and so in other cases.

FIRST or square-set of 16.

(Suffixes 0)

$\overset{u+u'}{\vartheta}$	$\overset{u-u'}{\vartheta}$	
0	0	$= XX' + YY' + ZZ' + WW'$
4	4	$= XX' - YY' + ZZ' - WW'$
8	8	$= XX' + YY' - ZZ' - WW'$
12	12	$= XX' - YY' - ZZ' + WW'$
1	1	$= YX' + XY' + WZ' + ZW'$
5	5	$= -YX' + XY' - WZ' + ZW'$
9	9	$= YX' + XY' - WZ' - ZW'$
13	13	$= -YX' + XY' + WZ' - ZW'$
2	2	$= ZX' + WY' + XZ' + YW'$
6	6	$= ZX' - WY' + XZ' - YW'$
10	10	$= -ZX' - WY' + XZ' + YW'$
14	14	$= -ZX' + WY' + XZ' - YW'$
3	3	$= WX' + ZY' + YZ' + XW'$
7	7	$= -WX' + ZY' - YZ' + XW'$
11	11	$= -WX' - ZY' + YZ' + XW'$
15	15	$= WX' - ZY' - YZ' + XW'$

91. SECOND set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

4	0	0	4	$= XX' + ZZ'$
12	8	8	12	$XX' - ZZ'$
5	1	1	5	$YX' + WZ'$
13	9	9	13	$YX' - WZ'$
6	2	2	6	$ZX' + XZ'$
14	10	10	14	$-ZX' + XZ'$
7	3	3	7	$WX' + YZ'$
15	11	11	15	$-WX' + YZ'$

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1)

4	0	0	4	$= YY' + WW'$
12	8	8	12	$YY' - WW'$
5	1	1	5	$XY' + ZW'$
13	9	9	13	$XY' - ZW'$
6	2	2	6	$WY' + YW'$
14	10	10	14	$-WY' + YW'$
7	3	3	7	$ZY' + XW'$
15	11	11	15	$-ZY' + XW'$

92. Third set of 16.

$$\tfrac{1}{2}\{ \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} \} \qquad \text{(Suffixes 2.)}$$

8	0	0	8	=	$XX'+YY'$
12	4	4	12		$XX'-YY'$
9	1	1	9		$YX'+XY'$
13	5	5	13		$-YX'+XY'$
10	2	2	10		$ZX'+WY'$
14	6	6	14		$ZX'-WY'$
11	3	3	11		$WX'+ZY'$
15	7	7	15		$-WX'+ZY'$

$$\tfrac{1}{2}\{ \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} \} \qquad \text{(Suffixes 2.)}$$

8	0	0	8	=	$ZZ'+WW'$
12	4	4	12		$ZZ'-WW'$
9	1	1	9		$WZ'+ZW'$
13	5	5	13		$-WZ'+ZW'$
10	2	2	10		$XZ'+YW'$
14	6	6	14		$XZ'-YW'$
11	3	3	11		$YZ'+XW'$
15	7	7	15		$-YZ'+XW'$

93. Fourth set of 16.

$$\tfrac{1}{2}\{ \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} \} \qquad \text{(Suffixes 3.)}$$

12	0	0	12	=	$XX'+WW'$
8	4	4	8		$XX'-WW'$
13	1	1	13		$YX'+ZW'$
9	5	5	9		$YX'-ZW'$
14	2	2	14		$ZX'+YW'$
10	6	6	10		$ZX'-YW'$
15	3	3	15		$WX'+XW'$
11	7	7	11		$WX'-XW'$

$$\tfrac{1}{2}\{ \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} \} \qquad \text{(Suffixes 3.)}$$

12	0	0	12	=	$YY'+ZZ'$
8	4	4	8		$-YY'+ZZ'$
13	1	1	13		$XY'+WZ'$
9	5	5	9		$-XY'+WZ'$
14	2	2	14		$WY'+XZ'$
10	6	6	10		$-WY'+XZ'$
15	3	3	15		$ZY'+YZ'$
11	7	7	11		$-ZY'+YZ'$

94. Fifth set of 16.

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

1	0	0	1	=	E+G	. E′+G′	+	F+H	. F′+H′
5	4	4	5		i.E−G	„	+	i.F−H	„
9	8	8	9		E+G	„	−	.F+H	„
13	12	12	13		i.E−G	„	−	i.F−H	„
3	2	2	3		F+H	„	+	E+G	„
7	6	6	7		i.F−H	„	+	i.E−G	„
11	10	10	11		− .F+H	„	+	E+G	„
15	14	14	15		−i.F−H	„	+	i.E−G	„

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

1	0	0	1	=	E−G	. E′−G′	+	F−H	. F′−H′
5	4	4	5		i.E+G	„	+	i.F+H	„
9	8	8	9		E−G	„	−	.F−H	„
13	12	12	13		i.E+G	„	−	i.F+H	„
3	2	2	3		F−H	„	+	E−G	„
7	6	6	7		i.F+H	„	+	i.E+G	„
11	10	10	11		− .F−H	„	+	E−G	„
15	14	14	15		−i.F+H	„	+	i.E+G	„

95. Sixth set of 16.

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

5	0	0	5	=	E−iG	. E′+iG′	+	F−iH	. F′+iH′
1	4	4	1		−i.E+iG	„	−	i.F+iH	„
8	13	13	8		E−iG	„	−	.F−iH	„
9	12	12	9		−i.E+iG	„	+	i.F+iH	„
7	2	2	7		F−iH	„	+	E−iG	„
3	6	6	3		−i.F+iH	„	−	i.E+iG	„
15	10	10	15		− .F−iH	„	+	E−iG	„
11	14	14	11		i.F+iH	„	−	i.E+iG	„

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{v-u'}{\vartheta}-\overset{v+u}{\vartheta}\,.\,\overset{u-u}{\vartheta}\}$ (Suffixes 1.)

5	0	0	5	=	E+iG	. E′−iG′	+	F+iH	. F′−iH′
1	4	4	1		−i.E−iG	„	−	i.F−iH	„
13	8	8	13		E+iG	„	−	.F+iH	„
9	12	12	9		−i.E−iG	„	+	i.F−iH	„
7	2	2	7		F+iH	„	+	E+iG	„
3	6	6	3		−i.F−iH	„	−	i.E−iG	„
15	10	10	15		− .F+iH	„	+	.E+iG	„
11	14	14	11		+i.F−iH	„	−	i.E−iG	„

96. Seventh set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

9	0	0	9	=	E+G	. E′+G′	+	F−H	. F′−H′
13	4	4	13		i.E−G	„	+	i.F+H	„
1	8	8	1		E+G	„	−	.F−H	„
5	12	12	5		i.E−G	„	−	i.F+H	„
11	2	2	11		F+H	„	+	E−G	„
15	6	6	15		i.F−H	„	+	i.E+G	„
3	10	10	3		F+H	„	−	.E−G	„
7	14	14	7		i.F−H	„	−	i.E+G	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

9	0	0	9	=	E−G	. E′−G′	+	F+H	. F′+H′
13	4	4	13		i.E+G	„	+	i.F−H	„
1	8	8	1		E−G	„	−	.F+H	„
5	12	12	5		i.E+G	„	−	i.F−H	„
11	2	2	11		F−H	„	+	E+G	„
15	6	6	15		i.F+H	„	+	i.E−G	„
3	10	10	3		F−H	„	−	.E+G	„
7	14	14	7		i F+H	„	−	i.E−G	„

97. Eighth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

13	0	0	13	=	E−iG	. E′+iG′	+	F+iH	. F′−iH′
9	4	4	9		−i.E+iG	„	−	i.F−iH	„
5	8	8	5		E−iG	„	−	.F+iH	„
1	12	12	1		−i E+iG	„	+	i.F−iH	„
15	2	2	15		F−iH	„	+	E+iG	„
11	6	6	11		−i.F+iH	„	−	i.E−iG	„
7	10	10	7		F−iH	„	−	.E+iG	„
3	14	14	3		−i F+iH	„	+	i.E−iG	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

13	0	0	13	=	E+iG	. E′−iG′	+	F−iH	. F′+iH′
9	4	4	9		−i.E−iG	„	−	i.F+iH	„
5	8	8	5		E+iG	„	−	.F−iH	„
1	12	12	1		−i E−iG	„	+	i.F+iH	„
15	2	2	15		F+iH	„	+	E−iG	„
11	6	6	11		−i.F−iH	„	−	i.E+iG	„
7	10	10	7		F+iH	„	−	.E−iG	„
3	14	14	3		−i.F−iH	„	+	i.E+iG	„

98. Ninth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

2	0	0	2	=	I+K	. I′+K′	+	J+L	. J′+L′
6	4	4	6		I+K	,,	−	.J+L	,,
10	8	8	10		i.I−K	,,	+	i.J−L	,,
14	12	12	14		i I−K	,,	−	i.J−L	,,
3	1	1	3		J+L	,,	+	I+K	,,
7	5	5	7		− .J+L	,,	−	.I+K	,,
11	9	9	11		i.J−L	,,	+	i.I−K	,,
15	13	13	15		−i J−L	,,	−	i.I−K	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

2	0	0	2	=	I−K	. I′−K′	+	J−L	. J′−L′
6	4	4	6		I−K	,,	−	.J−L	,,
10	8	8	10		i.I+K	,,	+	i.J+L	,,
14	12	12	14		i I+K	,	−	i J+L	,,
3	1	1	3		J−L	,,	+	I−K	,,
7	5	5	7		− .J−L	,,	+	I−K	,,
11	9	9	11		i.J+L	,,	+	i.I+K	,,
15	13	13	15		−i.J+L	,,	+	i.I+K	,,

99. Tenth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

6	0	0	6	=	I+K	. I′+K′	+	J−L	. J′−L′
2	4	4	2		I+K	,,	−	.J−L	,,
14	8	8	14		i.I−K	,,	+	i.J+L	,,
10	12	12	10		i.I−K	,,	−	i.J+L	,,
7	1	1	7		J+L	,,	+	I−K	,,
3	5	5	3		J+L	,,	−	.I−K	,,
15	9	9	15		i.J−L	,,	+	i.I+K	,,
11	13	13	11		i.J−L	,,	−	i.I+K	,,

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta}\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

6	0	0	6	=	I−K	. I′−K′	+	J+L	. J′+L′
2	4	4	2		I−K	,,	−	.J+L	,,
14	8	8	14		i.I+K	,,	+	i.J−L	,,
10	12	12	10		i.I+K	,,	−	i.J−L	,,
7	1	1	7		J−L	,,	+	I+K	,,
3	5	5	3		J−L	,,	−	.I+K	,,
15	9	9	15		i.J+L	,,	+	i.I−K	,,
11	13	13	11		i.J+L	,,	−	i.I−K	,,

100. Eleventh set of 16.

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

10	0	0	10	=	$I-iK$.	$I'+iK'$	+	$J-iL$.	$J'+iL'$
14	4	4	14		$I-iK$	"	−	$J-iL$	"
2	8	8	2		$-i\,I+iK$	"	−	$i.J+iL$	"
6	12	12	6		$-i.I+iK$	"	+	$i.J+iL$	"
11	1	1	11		$J-iL$	"	+	$I-iK$	"
15	5	5	15		$-.J-iL$	"	+	$I-iK$	"
3	9	9	3		$-i\,J+iL$	"	−	$i.I+iK$	"
7	13	13	7		$+i\,J+iL$	"	−	$i\,I+iK$	"

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u}{\vartheta}\}$ (Suffixes 2.)

10	0	0	10	=	$I+iK$.	$I'-iK'$	+	$J+iL$	$J'-iL'$
14	4	4	14		$I+iK$	"	−	$J+iL$	"
2	8	8	2		$-i.I-iK$	"	−	$i\,J-iL$	"
6	12	12	6		$-i.I-iK$	"	+	$i.J-iL$	"
11	1	1	11		$J+iL$	"	+	$I+iK$	"
15	5	5	15		$-\,J+iL$	"	+	$I+iK$	"
3	9	9	3		$-i.J-iL$	"	−	$i.I-iK$	"
7	13	13	7		$+i.J-iL$	"	−	$i.I-iK$	"

101. Twelfth set of 16.

$\frac{1}{2}\{\overset{u+u}{\vartheta}\,.\,\overset{u-u}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3)

14	0	0	14	=	$I-iK$.	$I'+iK'$	+	$J+iL$	$J'-iL'$
10	4	4	10		$I-iK$	"	−	$J+iL$	"
6	8	8	6		$-i.I+iK$	"	−	$i.J-iL$	"
2	12	12	2		$-i\,I+iK$	"	+	$i.J-iL$	"
15	1	1	15		$J-iL$	"	+	$I+iK$	"
11	5	5	11		$J-iL$	"	−	$I+iK$	"
7	9	9	7		$-i\,J+iL$	"	−	$i.I-iK$	"
3	13	13	3		$-i.J+iL$	"	+	$i.I-iK$	"

$\frac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

14	0	0	14	=	$I+iK$.	$I'-iK'$	+	$J-iL$.	$J'+iL'$
10	4	4	10		$I+iK$	"	−	$J-iL$	"
6	8	8	6		$-i\,I-iK$	"	−	$i.J+iL$	"
2	12	12	2		$-i.I-iK$	"	+	$i.J+iL$	"
15	1	1	15		$J+iL$	"	+	$I-iK$	"
11	5	5	11		$J+iL$	"	−	$.I-iK$	"
7	9	9	7		$-i.J-iL$	"	−	$i.I+iK$	"
3	13	13	3		$-i\,J-iL$	"	+	$i.I+iK$	"

102. Thirteenth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

3	0	0	3 =	M+Q	. M'+Q'	+	N+P	. N'+P'
7	4	4	7	i.M−Q	„	−	i.N−P	„
11	8	8	11	i.M−Q	„	+	i.N−P	„
15	12	12	15	− .M+Q	„	+	N+P	„
2	1	1	2	N+P	„	+	M+Q	„
6	5	5	6	−i.N−P	„	+	i.M−Q	„
10	9	9	10	i.N−P	„	+	i.M−Q	„
14	13	13	14	N+P	„	−	.M+Q	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 0.)

3	0	0	3 =	M−Q	. M'−Q'	+	N−P	. N'−P'
7	4	4	7	i.M+Q	„	−	i.N+P	„
11	8	8	11	i.M+Q	„	+	i.N+P	„
15	12	12	15	− .M−Q	„	+	N−P	„
2	1	1	2	N−P	„	+	M−Q	„
6	5	5	6	−i.N+P	„	+	i.M+Q	„
10	9	9	10	i.N+P	„	+	i.M+Q	„
14	13	13	14	N−P	„	−	.M−Q	„

103. Fourteenth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}+\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

7	0	0	7 =	M−iQ	. M'+iQ'	+	N+iP	. N'−iP'
3	4	4	3	−i.M+iQ	„	+	i.N−iP	„
15	8	8	15	i.M+iQ	„	+	i.N−iP	„
11	12	12	11	M−iQ	„	−	.N+iP	„
6	1	1	6	N−iP	„	+	M+iQ	„
2	5	5	2	−i.N−iP	„	+	i.M−iQ	„
14	9	9	14	+i.N+iP	„	+	i.M−iQ	„
10	13	13	10	N−iP	„	−	.M+iQ	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}-\overset{u+u'}{\vartheta}\,.\,\overset{u-u'}{\vartheta}\}$ (Suffixes 1.)

7	0	0	7 =	M+iQ	. M'−iQ'	+	N−iP	. N'+iP'
3	4	4	3	−i.M−iQ	„	+	i.N+iP	„
15	8	8	15	−i.M−iQ	„	+	i.N+iP	„
11	12	12	11	M+iQ	„	−	.N+iP	„
6	1	1	6	N+iP	„	+	M−iQ	„
2	5	5	2	−i.N−iP	„	+	i.M+iQ	„
14	9	9	14	+i.N−iP	„	+	i.M+iQ	„
10	13	13	10	N+iP	„	−	.M−iQ	„

104. Fifteenth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u}{\vartheta}\}$ (Suffixes 2.)

11	0	0	11	=	$M-iQ$	$.\ M'+iQ'$	+	$N-iP$	$.\ N'+iP'$
15	4	4	15		$i.M+iQ$	„	−	$i.N+iP$	„
3	8	8	3		$-i.M+iQ$	„	−	$i.N+iP$	„
7	12	12	7		$M-iQ$	„	−	$.N-iP$	„
10	1	1	10		$N-iP$	„	+	$M-iQ$	„
14	5	5	14		$-i.N+iP$	„	+	$i.M+iQ$	„
2	9	9	2		$-i.N+iP$	„	−	$i.M+iQ$	„
6	13	13	6		$-\ .N-iP$	„	+	$M-iQ$	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta}\}$ (Suffixes 2.)

11	0	0	11	=	$M+iQ$	$.\ M'-iQ'$	+	$N+iP$	$.\ N'-iP'$
15	4	4	15		$i.M-iQ$	„	−	$i.N-iP$	„
3	8	8	3		$-i.M-iQ$	„	−	$i.N-iP$	„
7	12	12	7		$M+iQ$	„	−	$.N+iP$	„
10	1	1	10		$N+iP$	„	+	$M+iQ$	„
14	5	5	14		$-i.N-iP$	„	+	$i.M-iQ$	„
2	9	9	2		$-i.N-iP$	„	−	$i.M-iQ$	„
6	13	13	6		$-\ .N+iP$	„	+	$M+iQ$	„

105. Sixteenth set of 16.

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} + \overset{u+u'}{\vartheta} \,.\, \overset{u-u}{\vartheta}\}$ (Suffixes 3.)

15	0	0	15	=	$M-Q$	$.\ M'-Q'$	+	$N+P$	$.\ N'+P'$
11	4	4	11		$-i.M+Q$	„	+	$i.N-P$	„
7	8	8	7		$-i.M+Q$	„	−	$i.N-P$	„
3	12	12	3		$-\ .M-Q$	„	+	$N+P$	„
14	1	1	14		$N-P$	„	+	$M+Q$	„
10	5	5	10		$-i.N+P$	„	+	$i.M-Q$	„
6	9	9	6		$-i.N+P$	„	−	$i.M-Q$	„
2	13	13	2		$-\ .N-P$	„	+	$M+Q$	„

$\tfrac{1}{2}\{\overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta} - \overset{u+u'}{\vartheta} \,.\, \overset{u-u'}{\vartheta}\}$ (Suffixes 3.)

15	0	0	15	=	$M+Q$	$.\ M'+Q'$	+	$N-P$	$.\ N'-P'$
11	4	4	11		$-i.M-Q$	„	+	$i.N+P$	„
7	8	8	7		$-i.M-Q$	„	−	$i.N+P$	„
3	12	12	3		$-\ .M+Q$	„	+	$N-P$	„
14	1	1	14		$N+P$	„	+	$M-Q$	„
10	5	5	10		$-i.N-P$	„	+	$i.M+Q$	„
6	9	9	6		$-i.N-P$	„	−	$i.M+Q$	„
2	13	13	2		$-\ .N+P$	„	+	$M-Q$	„

106. In the square set, writing $u'=v'=0$, and $\alpha, \beta, \gamma, \delta$ for X', Y', Z', W'; also slightly altering the arrangement,

the system becomes

u			X	Y	Z	W
ϑ^2	0	=	α	β	γ	δ
	4	=	α	$-\beta$	γ	$-\delta$
	8	=	α	β	$-\gamma$	$-\delta$
	12	=	α	$-\beta$	$-\gamma$	δ
	1	=	β	α	δ	γ
	5	=	β	$-\alpha$	δ	$-\gamma$
	9	=	β	α	$-\delta$	$-\gamma$
	13	=	β	$-\alpha$	$-\delta$	γ
	2	=	γ	δ	α	β
	6	=	γ	$-\delta$	α	$-\beta$
	10	=	γ	δ	$-\alpha$	$-\beta$
	14	=	γ	$-\delta$	$-\alpha$	β
	3	=	δ	γ	β	α
	7	=	δ	$-\gamma$	β	$-\alpha$
	11	=	δ	γ	$-\beta$	$-\alpha$
	15	=	δ	$-\gamma$	$-\beta$	α

and further writing herein $u=0$, $v=0$ it becomes

0						c^2
ϑ^2	0	=	$\alpha^2+\beta^2+\gamma^2+\delta^2$		=	0
	4	=	$\alpha^2-\beta^2+\gamma^2-\delta^2$		=	4
	8	=	$\alpha^2-\beta^2-\gamma^2-\delta^2$		=	8
	12	=	$\alpha^2-\beta^2-\gamma^2+\delta^2$		=	12
	1	=	$2(\alpha\beta+\gamma\delta)$		=	1
	5			0		
	9	=	$2(\alpha\beta-\gamma\delta)$		=	9
	13			0		
	2	=	$2(\alpha\gamma+\beta\delta)$		=	2
	6	=	$2(\alpha\gamma-\beta\delta)$		=	6
	10			0		
	14			0		
	3	=	$2(\alpha\delta+\beta\gamma)$		=	3
	7			0		
	11			0		
	15	=	$2(\alpha\delta-\beta\gamma)$		=	15

viz.: this last is the before-mentioned system of equations giving the values of the 10 zero-functions c in terms of the four constants $\alpha, \beta, \gamma, \delta$.

107. The system first obtained is a system of 16 equations

$$\vartheta_0^2(u, v)=\alpha X+\beta Y+\gamma Z+\delta W, \text{ \&c.}$$

showing that the squares of the theta-functions are each of them a linear function of the four quantities X, Y, Z, W. If the functions on the right hand side were independent (asyzygetic) linear functions of (X, Y, Z, W) it would follow that any four (selected at pleasure) of the squared theta-functions were linearly independent, and that we could in terms of these four express linearly each of the remaining 12 squared functions. But this is not so; the form of the linear functions of (X, Y, Z, W) is such that we can (and that in 16 different ways) select out of the 16 linear functions six functions, such that any four of them are connected by a linear equation; and there are consequently 16 hexads of squared theta-functions, such that any four out of the same hexad are connected by a linear relation. The hexads are shown by the foregoing "Table of the 16 KUMMER hexads."

108. The *à posteriori* verification is immediately effected; taking for instance the first column, the equations are

	$\vartheta^2 u$		X	Y	Z	W
A	11	=	δ	γ	$-\beta$	$-\alpha$,
B	7		δ	$-\gamma$	β	$-\alpha$,
AB	6		γ	$-\delta$	α	$-\beta$,
CD	2		γ	δ	α	δ,
CE	1		β	α	δ	γ,
DE	9		β	α	δ	$-\gamma$.

viz.: it should thence follow that there is a linear relation between any four of the six squared functions 11, 7, 6, 2, 1, 9: and it is accordingly seen that this is so. It further appears that in the several linear relations, the coefficients (obtained in the first instance as functions of α, β, γ, δ) are in fact the 10 constants c: the 15 relations connecting the several systems of four out of the six squared functions are given in the table.

Read

$$\cdot \quad c_6^2\vartheta_6^2 - c_2^2\,\vartheta_2^2 + c_1^2\,\vartheta_1^2 - c_9^2\vartheta_9^2 = 0,$$

$$c_6^2\vartheta_{11}^2 \quad \cdot \quad + c_{15}^2\vartheta_2^2 - c_{12}^2\vartheta_1^2 + c_4^2\vartheta_9^2 = 0, \quad \&c.$$

109.

ϑ^2	11	7	6	2	1	9	= 0
c^2			6	− 2	1	− 9	
	6			+15	−12	+ 4	
	− 2		−15		+ 8	− 0	
	1		+12	− 8		+ 3	
	− 9		− 4	+ 0	− 3		
		6		3	− 0	+ 8	
		− 2	− 3		+ 4	−12	
		1	+ 0	− 4		−15	
		− 9	− 8	+12	+15		
	−15	+ 3	+ 2	− 6			
	−12	+ 0	+ 1		− 6		
	− 4	+ 8	+ 9			− 6	
	− 3	+15			+ 9	− 1	
	− 0	+12		+ 9		− 2	
	− 8	+ 4		+ 1	− 2		

110. The first set of 16 equations is the square-set, which has been already considered. If in each of the other sets of 16 equations we write in like manner $u'=0$, each set in fact reduces itself to eight equations; sets 2, 3, 4 give thus $8+8+8$, $=24$ equations; sets 5 to 8, 9 to 12, and 13 to 15, give each $8+8+8+8$, $=32$ equations; or we have sets of 24, 32, 32, 32, together 120 equations, the number being of course one half of $256-16$, the number of equations after deducting the 16 equations of the square set.

111. The first set, 24 equations.

This is derived from the second, third, and fourth sets, each of 16 equations, by writing therein $u'=0$. Taking α_1, β_1, γ_1, δ_1 for the zero-functions corresponding to X_1, Y_1, Z_1, W_1, then on writing $u'=0$, X_1', Y_1', Z_1', W_1' become α_1, β_1, γ_1, δ_1. In the second set of 16 equations, the first equations thus are

$$\begin{array}{ll} \vartheta_4 u \,.\, \vartheta_0 u = \alpha_1 X_1 + \gamma_1 Z_1, & \qquad 0 = \beta_1 Y_1 + \delta_1 W_1, \\ \vartheta_{12} u \;\; \vartheta_8 u = \alpha_1 X_1 - \gamma_1 Z_1, & \qquad 0 = \beta_1 Y_1 - \delta_1 W_1, \\ \vdots & \qquad \vdots \end{array}$$

viz, the equations of the column require that, and are all satisfied if, $\beta_1=0$, $\delta_1=0$: hence the zero functions are α_1, 0, γ_1, 0; and this being so we have only the equations of the first column. And similarly as regards the third and fourth sets; the zero values corresponding to

$$\text{are} \qquad \begin{array}{cccc|cccc|cccc} X_1, & Y_1, & Z_1, & W_1 & X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3 \\ \alpha_1 & 0 & \gamma_1 & 0 & \alpha_2 & \beta_2 & 0 & 0 & \alpha_3 & 0 & 0 & \delta_3; \end{array}$$

and we have in all 8+8+8, =24 equations. These are

		(Suffixes 1.)					(Suffixes 2)					(Suffixes 3)		
ϑu	ϑu		X	Z	ϑu .	ϑu		X	Y	ϑu .	ϑu		X	W
4	0	=	α	γ	8	0	=	α	β	12	0	=	α	δ
12	8	=	α	$-\gamma$	12	4	=	α	$-\beta$	8	4	=	α	$-\delta$
6	2	=	γ	α	9	1	=	β	α	15	3	=	δ	α
14	10	=	γ	$-\alpha$	13	5	=	β	$-\alpha$	11	7	=	$-\delta$	α
			Y	W				Z	W				Y	Z
5	1	=	α	γ	10	2	=	α	β	13	1	=	α	δ
13	9	=	α	$-\gamma$	14	6	=	α	$-\beta$	9	5	=	α	$-\delta$
7	3	=	γ	α	11	3	=	β	α	14	2	=	δ	α
15	11	=	γ	$-\alpha$	15	7	=	β	$-\alpha$	10	6	=	$-\delta$	α

$\vartheta 0$	$\vartheta 0$			$\vartheta 0$	$\vartheta 0$			$\vartheta 0$	$\vartheta 0$		
4	0	=	$\alpha^2+\gamma^2$	8	0	=	$\alpha^2+\beta^2$	12	0	=	$\alpha^2+\delta^2$
12	8	=	$\alpha^2-\gamma^2$	12	4	=	$\alpha^2-\beta^2$	8	4	=	$\alpha^2-\delta^2$
6	2	=	$2\alpha\gamma$	9	1	=	$2\alpha\beta$	15	3	=	$2\alpha\delta$

112. THE second set, 32 equations.

To exhibit these in a convenient form I alter the notation, viz., I write

$$\begin{array}{cccc|cccc|}
E+G, & i(E-G), & (F+H), & i(F-H) & E_1+iG_1, & E_1-iG_1, & F_1+iH_1, & F_1-iH_1 \\
= X, & Y, & Z, & W & X_1, & Y_1, & Z_1, & W_1
\end{array}$$

$$\begin{array}{cccc|cccc}
(E_2+G_2), & i(E_2-G_2), & (F_2+H_2), & i(F_2-H_2) & E_3+iG_3, & E_3-iG_3, & F_3+iH_3, & F_3-iH_3 \\
= X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3,
\end{array}$$

so that as regards the present set of equations, X, Y, X, &c., signify as just mentioned. And this being so the corresponding zero-values are

$$\alpha,\ 0,\ \gamma,\ 0\ \mid\ \alpha_1,\ 0,\ \gamma_1,\ 0\ \mid\ \alpha_2,\ 0,\ 0,\ \delta_2\ \mid\ \alpha_3,\ 0,\ 0,\ \delta_3.$$

The equations then are

(Suffixes 0.)	(Suffixes 1)	(Suffixes 2.)	(Suffixes 3.)
$\vartheta u\,.\,\vartheta u\quad \overbrace{X\quad Z}$	$\vartheta u\,.\,\vartheta u\quad \overbrace{X\quad Z}$	$\vartheta u\,.\,\vartheta u\quad \overbrace{X\quad W}$	$\vartheta u\,.\,\vartheta u\quad \overbrace{X\quad W}$
$1\quad 0 = \alpha\quad \gamma$	$1\quad 4 = -i\alpha\quad -i\gamma$	$9\quad 0 = \alpha\quad -\delta$	$9\quad 4 = -i\alpha\quad -i\delta$
$9\quad 8 = \alpha\quad -\gamma$	$9\quad 12 = -i\alpha\quad +i\gamma$	$1\quad 8 = \alpha\quad \delta$	$1\quad 12 = -i\alpha\quad +i\delta$
$3\quad 2 = \gamma\quad \alpha$	$3\quad 6 = -i\gamma\quad -i\alpha$	$15\quad 6 = \delta\quad \alpha$	$15\quad 2 = \delta\quad \alpha$
$11\quad 10 = \gamma\quad -\alpha$	$11\quad 14 = -i\gamma\quad +i\alpha$	$7\quad 14 = -\delta\quad \alpha$	$7\quad 10 = -\delta\quad \alpha$
$\overbrace{Y\quad W}$	$\overbrace{Y\quad W}$	$\overbrace{Y\quad Z}$	$\overbrace{Y\quad Z}$
$5\quad 4 = \alpha\quad \gamma$	$5\quad 0 = \alpha\quad \gamma$	$13\quad 14 = \alpha\quad \delta$	$13\quad 0 = \alpha\quad \delta$
$13\quad 12 = \alpha\quad -\gamma$	$13\quad 8 = \alpha\quad -\gamma$	$5\quad 12 = \alpha\quad -\delta$	$5\quad 8 = \alpha\quad -\delta$
$7\quad 6 = \gamma\quad \alpha$	$7\quad 2 = \gamma\quad \alpha$	$11\quad 2 = -\delta\quad \alpha$	$11\quad 6 = -i\delta\quad -i\alpha$
$15\quad 14 = \gamma\quad -\alpha$	$15\quad 10 = \gamma\quad -\alpha$	$3\quad 10 = \delta\quad \alpha$	$3\quad 14 = i\delta\quad -i\alpha$
$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$	$\vartheta 0\,.\,\vartheta 0$
$1\quad 0 = \alpha^2+\gamma^2$	$1\quad 4 = -i(\alpha^2+\gamma^2)$	$9\quad 0 = \alpha^2-\delta^2$	$9\quad 4 = -i(\alpha^2+\delta^2)$
$9\quad 8 = \alpha^2-\gamma^2$	$9\quad 12 = -i(\alpha^2-\gamma^2)$	$1\quad 8 = \alpha^2+\delta^2$	$1\quad 12 = -i(\alpha^2-\delta^2)$
$3\quad 2 = 2\alpha\gamma$	$3\quad 6 = -\ 2i\alpha\gamma$	$15\quad 6 = 2\alpha\delta$	$15\quad 2 = 2\alpha\delta$

113. THIRD set, 32 equations.

We again change the notation, writing

$$\begin{array}{l|l|}
\mathrm{I}+\mathrm{K},\ i(\mathrm{I}-\mathrm{K}),\ \mathrm{J}+\mathrm{L},\ i(\mathrm{J}-\mathrm{L}) & \mathrm{I}_1+\mathrm{K}_1,\ i(\mathrm{I}_1-\mathrm{K}_1),\ (\mathrm{J}_1+\mathrm{L}_1),\ i(\mathrm{J}_1-\mathrm{L}_1) \\
=\ \mathrm{X},\quad \mathrm{Y},\quad \mathrm{Z},\quad \mathrm{W} & \mathrm{X}_1,\quad \mathrm{Y}_1,\quad \mathrm{Z}_1,\quad \mathrm{W}_1
\end{array}$$

$$\begin{array}{l|l}
\mathrm{I}_2+i\mathrm{K}_2,\ \mathrm{I}_2-i\mathrm{K}_2,\ \mathrm{J}_2+i\mathrm{L}_2,\ \mathrm{J}_2-i\mathrm{L}_2 & \mathrm{I}_3+i\mathrm{K}_3,\ \mathrm{I}_3-i\mathrm{K}_3,\ \mathrm{J}_3+i\mathrm{L}_3,\ \mathrm{J}_3-i\mathrm{L}_3 \\
=\ \mathrm{X}_2,\quad \mathrm{Y}_2,\quad \mathrm{Z}_2,\quad \mathrm{W}_2 & \mathrm{X}_3,\quad \mathrm{Y}_3,\quad \mathrm{Z}_3,\quad \mathrm{W}_3
\end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \ \big|\ \alpha_1,\ 0,\ 0,\ \delta_1 \ \big|\ \alpha_2,\ 0,\ \gamma_2,\ 0 \ \big|\ \alpha_3,\ 0,\ 0,\ \delta_3$$

Then equations are

(Suffixes 0.)			(Suffixes 1.)			(Suffixes 2.)			(Suffixes 3.)		
$\vartheta u\,.\,\vartheta u$	X	Z	$\vartheta u\,.\,\vartheta u$	X	W	$\vartheta u\,.\,\vartheta u$	X	Z	$\vartheta u\,.\,\vartheta u$	X	W
2 0 =	α	γ	6 0 =	α	$-\delta$	2 8 =	$-i\alpha$	$-i\gamma$	6 8 =	$-i\alpha$	$-i\delta$
6 4 =	α	$-\gamma$	2 4 =	α	δ	6 12 =	$-i\alpha$	$+i\gamma$	2 12 =	$-i\alpha$	$i\delta$
3 1 =	γ	α	15 9 =	δ	α	3 9 =	$-i\gamma$	$-i\alpha$	15 1 =	δ	α
7 5 =	γ	$-\alpha$	11 13 =	$-\delta$	α	7 13 =	$-i\gamma$	$+i\alpha$	11 5 =	$-\delta$	α
	Y	W		Y	Z		Y	W		Y	Z
10 8 =	α	γ	14 8 =	α	δ	10 0 =	α	γ	14 0 =	α	δ
14 12 =	α	$-\gamma$	10 12 =	α	$-\delta$	14 4 =	α	$-\gamma$	10 4 =	α	$-\delta$
11 9 =	γ	α	7 1 =	$-\delta$	α	11 1 =	γ	α	7 9 =	$-i\delta$	$-i\alpha$
15 13 =	γ	$-\alpha$	3 5 =	δ	α	15 5 =	γ	$-\alpha$	3 13 =	$i\delta$	$-i\alpha$
$\vartheta 0\,.\,\vartheta 0$			$\vartheta 0\,.\,\vartheta 0$			$\vartheta 0\,.\,\vartheta 0$			$\vartheta 0\,.\,\vartheta 0$		
2 0 =	$\alpha^2+\gamma^2$		6 0 =	$\alpha^2-\delta^2$		2 8 =	$-i(\alpha^2+\gamma^2)$		6 8 =	$-i(\alpha^2+\delta^2)$	
6 4 =	$\alpha^2-\gamma^2$		2 4 =	$\alpha^2+\delta^2$		6 12 =	$-i(\alpha^2-\gamma^2)$		2 12 =	$-i(\alpha^2-\delta^2)$	
3 1 =	$2\alpha\gamma$		15 9 =	$2\alpha\delta$		3 9 =	$-2i\alpha\gamma$		15 1 =	$2\alpha\delta$	

114. FOURTH set, 32 equations.

Again changing the notation we write

$$\begin{array}{rcccc|cccc} & M_2+iQ_2, & M_2-iQ_2, & N_2+iP_2, & N_2-iP_2 & M_3+Q_3, & i(M_3-Q_3), & N_3+P_3, & i(N_3-P_3) \\ = & X, & Y, & Z, & W & X_1, & Y_1, & Z_1, & W_1, \end{array}$$

$$\begin{array}{rcccc|cccc} & M+Q, & i(M-Q), & N+P, & i(N-P) & M_1+iQ_1, & M_1-iQ_1, & N_1+iP_1, & N_1-iP_1 \\ = & X_2, & Y_2, & Z_2, & W_2 & X_3, & Y_3, & Z_3, & W_3 \end{array}$$

the zero values being

$$\alpha,\ 0,\ \gamma,\ 0 \ \big|\ \alpha_1,\ 0,\ 0_1,\ \delta \ \big|\ \alpha_2,\ 0,\ \gamma_2,\ 0 \ \big|\ 0,\ \beta_3,\ \gamma_3,\ 0$$

and the equations then are

(Suffixes 0)

ϑu	ϑu		X	Z
0	3	=	α	γ
15	12	=	$-\alpha$	γ
2	1	=	γ	α
14	13	=	$-\gamma$	α
			Y	W
4	7	=	α	$-\gamma$
8	11	=	α	γ
6	5	=	γ	$-\alpha$
10	9	=	γ	α
$\vartheta 0$	$\vartheta 0$			
0	3	=		$\alpha^2+\gamma^2$
15	12	=		$-(\alpha^2-\gamma^2)$
2	1	=		$2\alpha\gamma$

(Suffixes 1)

ϑu	ϑu		X	W
3	4	=	$-i\alpha$	$i\delta$
15	8	=	$i\alpha$	$i\delta$
6	1	=	δ	α
10	13	=	$-\delta$	α
			Y	Z
7	0	=	α	δ
11	12	=	α	$-\delta$
2	5	=	$i\delta$	$-i\alpha$
14	9	=	$i\delta$	$i\alpha$
$\vartheta 0$	$\vartheta 0$			
3	4	=		$-i(\alpha^2-\delta^2)$
15	8	=		$i(\alpha^2+\delta^2)$
6	1	=		$2\alpha\delta$

(Suffixes 2)

ϑu	ϑu		X	Z
15	4	=	$i\alpha$	$-i\gamma$
3	8	=	$-i\alpha$	$-i\gamma$
14	5	=	$i\gamma$	$-i\alpha$
2	9	=	$-i\gamma$	$-i\alpha$
			Y	W
11	0	=	α	γ
7	12	=	α	$-\gamma$
10	1	=	γ	α
6	13	=	γ	$-\alpha$
$\vartheta 0$	$\vartheta 0$			
15	4	=		$i(\alpha^2-\gamma^2)$
3	8	=		$-i(\alpha^2+\gamma^2)$
2	9	=		$-\ 2i\alpha\gamma$

(Suffixes 3)

ϑu	ϑu		Y	Z
15	0	=	$-\beta$	γ
3	12	=	β	γ
10	5	=	γ	$-\beta$
6	9	=	$-\gamma$	$-\beta$
			X	W
11	4	=	$-\beta$	γ
7	8	=	$-\beta$	$-\gamma$
14	1	=	γ	$-\beta$
2	13	=	γ	β
$\vartheta 0$	$\vartheta 0$			
15	0	=		$-(\beta^2-\gamma^2)$
3	12	=		$\beta^2+\gamma^2$
6	9	=		$-\ 2\beta\gamma$

115. It will be noticed that the pairs of theta-functions which present themselves in these equations are the same as in the foregoing "Table of the 120 pairs." And the equations show that the four products, each of a pair of theta-functions, belonging to the upper half or to the lower half of any column of the table, are such that any three of the four products are connected by a linear equation. The coefficients of these linear relations are, in fact, functions such as the $\alpha^2+\delta^2$, $\alpha^2-\delta^2$, $2\alpha\delta$ written down at the foot of the several systems of eight equations, and they are consequently products each of two zero-functions c.

Thus (see "The first set, 24 equations") we have

$\vartheta u\,.\,\vartheta u$			(Suffixes 3) X	W	$\vartheta u\,.\,\vartheta u$			(Suffixes 3) Y	Z	$\vartheta 0\,.\,\vartheta 0$		(Suffixes 3)
4	8	=	α	$-\delta$	5	9	=	α	$-\delta$	4	8	$=\alpha^2-\delta^2$
0	12	=	α	δ	1	13	=	α	δ	0	12	$=\alpha+\delta^2$
3	15	=	δ	α	2	14	=	δ	α	15	3	$=2\alpha\delta$
7	11	=	$-\delta$	α	6	10	=	$-\delta$	α			

116. In the left hand four of these, omitting successively the first, second, third, and fourth equation, and from the remaining three eliminating the X_3 and W_3, we write down, almost mechanically,

$$\begin{array}{cc|cccc|}
\vartheta u & .\ \vartheta u & & & & \\
4 & 8 & . & +2\alpha\delta, & -\delta^2-\alpha^2, & \alpha^2-\delta^2 \\
0 & 12 & -2\alpha\delta, & & -\delta^2+\alpha^2, & \alpha^2+\delta^2 \\
3 & 15 & \alpha^2+\delta^2, & \delta^2-\alpha^2, & . & 2\alpha\delta \\
7 & 11 & -\alpha^2+\delta^2, & \delta^2+\alpha^2, & -2\alpha\delta &
\end{array}$$

and thence derive the first of the next following system of equations; read

$$\begin{array}{rrrr}
 & c_3c_{15}\vartheta_0\vartheta_{12} & -c_0c_{12}\vartheta_3\vartheta_{15} & +c_4c_8\,\vartheta_7\vartheta_{11}=0, \\
-c_3c_{15}\vartheta_4\vartheta_8 & & +c_4c_8\,\vartheta_3\vartheta_{15} & -c_0c_{12}\vartheta_7\vartheta_{11}=0, \\
c_0c_{12}\vartheta_4\vartheta_8 & -c_4c_8\,\vartheta_0\vartheta_{12} & & +c_3c_{15}\vartheta_7\vartheta_{11}=0, \\
-c_4c_8\,\vartheta_4\vartheta_8 & +c_0c_{12}\vartheta_0\vartheta_{12} & -c_3c_{15}\vartheta_3\vartheta_{15} & =0,
\end{array}$$

where the theta-functions have the arguments u, v.

Observe that on writing herein $u=0$, $v=0$, the first three equations become each of them identically $0=0$; the fourth equation becomes $-c_4{}^2c_8{}^2+c_0{}^2c_{12}{}^2-c_3{}^2c_{15}{}^2=0$, which is one of the relations between the c's, and which serves as a verification.

But in the right hand system, on writing $u=v=0$, each of the four equations becomes identically $0=0$.

117. The equations are

$$\begin{array}{c|cccc} \vartheta & 4.8 & 0.12 & 3.15 & 7.11 \\ \hline c & & 3.15 & -0.12 & 4.8 \\ & -3.15 & & -4.8 & -0.12 \\ & 0.12 & -4.8 & & 3.15 \\ & -4.8 & 0.12 & -3.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 5.9 & 1.13 & 2.14 & 6.10 \\ \hline c & & 3.15 & -0.12 & 4.8 \\ & -3.15 & & 4.8 & -0.12 \\ & 0.12 & -4.8 & & 3.15 \\ & -4.8 & -0.12 & -3.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 6.8 & 2.12 & 1.15 & 5.11 \\ \hline c & & 1.15 & -2.12 & 6.8 \\ & -1.15 & & 6.8 & -2.11 \\ & 2.12 & -6.8 & & 1.15 \\ & -6.8 & 2.11 & -1.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 7.9 & 3.13 & 0.14 & 4.10 \\ \hline c & & 1.15 & -2.12 & 6.8 \\ & -1.15 & & 6.8 & -2.12 \\ & 2.12 & -6.8 & & 1.15 \\ & -6.8 & 2.12 & -1.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 0.6 & 2.4 & 9.15 & 11.13 \\ \hline c & & 9.15 & -2.4 & 0.6 \\ & -9.15 & & 0.6 & -2.4 \\ & 2.4 & -0.6 & & 9.15 \\ & -0.6 & 2.4 & -9.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 1.7 & 3.5 & 8.14 & 10.12 \\ \hline c & & 9.15 & -2.4 & 0.6 \\ & -9.15 & & 0.6 & -2.4 \\ & 2.4 & -0.6 & & 9.15 \\ & -0.6 & 2.4 & -9.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 3.6 & 1.4 & 9.12 & 14.11 \\ \hline c & & 9.12 & -1.4 & 3.6 \\ & -9.12 & & 3.6 & -1.4 \\ & 1.4 & -3.6 & & 9.12 \\ & -3.6 & 1.4 & -9.12 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 2.7 & 3.5 & 0.5 & 8.13 \\ \hline c & & 9.12 & -1.4 & 3.6 \\ & -9.12 & & 3.6 & -1.4 \\ & 1.4 & -3.6 & & 9.12 \\ & -3.6 & 1.4 & -9.12 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 8.9 & 0.1 & 2.3 & 10.11 \\ \hline c & & -2.3 & 0.1 & 8.9 \\ & 2.3 & & -8.9 & -0.1 \\ & -0.1 & 8.9 & & 2.3 \\ & -8.9 & 0.1 & -2.3 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 12.13 & 4.5 & 6.7 & 14.15 \\ \hline c & & -2.3 & 0.1 & 8.9 \\ & 2.3 & & -8.9 & -0.1 \\ & -0.1 & 8.9 & & 2.3 \\ & -8.9 & 0.1 & -2.3 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 4.6 & 0.2 & 1.3 & 5.7 \\ \hline c & & -1.3 & 2.0 & 4.6 \\ & 1.3 & & -4.6 & -0.2 \\ & -0.2 & 4.6 & & 1.3 \\ & -4.6 & 0.2 & -1.3 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 9.11 & 13.15 & 12.14 & 8.10 \\ \hline c & & -1.3 & 0.2 & 4.6 \\ & 1.3 & & -4.6 & -0.2 \\ & -0.2 & -4.6 & & 1.3 \\ & 4.6 & 0.2 & -1.3 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 6.12 & 2.8 & 3.9 & 7.13 \\ \hline c & & 3.9 & -2.8 & -6.12 \\ & -3.9 & & 6.12 & 2.8 \\ & 2.8 & -6.12 & & -3.9 \\ & 6.12 & -2.8 & 3.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 1.11 & 5.15 & 4.14 & 0.10 \\ \hline c & & 3.9 & -2.8 & 6.12 \\ & -3.9 & & -6.12 & 2.8 \\ & 2.8 & 6.12 & & -3.9 \\ & -6.12 & -2.8 & 3.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 6.15 & 1.8 & 0.9 & 7.14 \\ \hline c & & 0.9 & -1.8 & -6.15 \\ & -0.9 & & 6.15 & 1.8 \\ & 1.8 & -6.15 & & -0.9 \\ & 6.15 & -1.8 & 0.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 2.11 & 5.12 & 4.13 & 3.10 \\ \hline c & & 0.9 & -1.8 & 6.15 \\ & -0.9 & & -6.15 & 1.8 \\ & 1.8 & 6.15 & & -0.9 \\ & -6.15 & -1.8 & 0.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 4.9 & 1.12 & 2.15 & 7.10 \\ \hline c & & 2.15 & -1.12 & 4.9 \\ & -2.15 & & 4.9 & -1.12 \\ & 1.12 & -4.9 & & 2.15 \\ & -4.9 & 1.12 & -2.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 0.13 & 5.8 & 6.11 & 3.14 \\ \hline c & & 2.15 & 1.12 & -4.9 \\ & -2.15 & & -4.9 & 1.12 \\ & -1.12 & 4.9 & & 2.15 \\ & 4.9 & -1.12 & -2.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 4.12 & 0.8 & 1.9 & 5.13 \\ \hline c & & -1.9 & 0.8 & 4.12 \\ & 1.9 & & -4.12 & -0.8 \\ & -0.8 & 4.12 & & 1.9 \\ & -4.12 & 0.8 & -1.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 3.11 & 7.15 & 6.14 & 2.10 \\ \hline c & & 1.9 & -0.8 & 4.12 \\ & -1.9 & & -4.12 & 0.8 \\ & 0.8 & 4.12 & & -1.9 \\ & -4.12 & -0.8 & 1.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 4.15 & 3.8 & 2.9 & 5.14 \\ \hline c & & -2.9 & 3.8 & 4.15 \\ & 2.9 & & -4.15 & -3.8 \\ & -3.8 & 4.15 & & 2.9 \\ & -4.15 & 3.8 & -2.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 0.11 & 7.12 & 6.13 & 1.10 \\ \hline c & & -2.9 & 3.8 & -4.15 \\ & 2.9 & & 4.15 & -3.8 \\ & -3.8 & -4.15 & & 2.9 \\ & 4.15 & 3.8 & -2.9 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 6.9 & 3.12 & 0.15 & 5.10 \\ \hline c & & -0.15 & 3.12 & -6.9 \\ & 0.15 & & -6.9 & 3.12 \\ & -3.12 & 6.9 & & -0.15 \\ & 6.9 & -3.12 & 0.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 2.13 & 7.8 & 4.11 & 1.14 \\ \hline c & & 0.15 & 3.12 & -6.9 \\ & -0.15 & & -6.9 & 3.12 \\ & -3.12 & 6.9 & & 0.15 \\ & 6.9 & -3.12 & -0.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 12.15 & 0.3 & 1.2 & 13.14 \\ \hline c & & 1.2 & -0.3 & -12.15 \\ & -1.2 & & 12.15 & 0.3 \\ & 0.3 & -12.15 & & -1.2 \\ & 12.15 & -0.3 & 1.2 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 8.11 & 4.7 & 5.6 & 9.10 \\ \hline c & & 1.2 & -0.3 & 12.15 \\ & -1.2 & & -12.15 & 0.3 \\ & 0.3 & 12.15 & & -1.2 \\ & -12.15 & -0.3 & 1.2 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 1.6 & 3.4 & 8.15 & 10.13 \\ \hline c & & 8.15 & -3.4 & 1.6 \\ & -8.15 & & 1.6 & -3.4 \\ & 3.4 & -1.6 & & 8.15 \\ & -1.6 & 3.4 & -8.15 & \end{array} =0,$$

$$\begin{array}{c|cccc} \vartheta & 2.5 & 0.7 & 11.12 & 9.14 \\ \hline c & & -8.15 & -3.4 & 1.6 \\ & 8.15 & & 1.6 & -3.4 \\ & 3.4 & -1.6 & & -8.15 \\ & 8.15 & 3.4 & 8.15 & \end{array} =0,$$

ϑ	2.6	0.4	8.12	10 14	=0,
c		−8 12	0 4	−2.6	
	8.12		−2 6	0.4	
	−0.4	2.6		−8.12	
	2 6	−0 4	8.12		

ϑ	1.5	3 7	11.15	9 13	=0.
c		−8 12	−0.4	2 6	
	8.12		2.6	−0 4	
	0 4	−2 6		−8.12	
	−2.6	0.4	8 12		

118. The foregoing equations may be verified, and it is interesting to verify them, by means of the approximate values of the functions: thus for one of the equations we have

$$\begin{array}{l|l}
c_3c_{15}\vartheta_0\vartheta_{12}\ \textit{i.e.} & (2\Lambda+2\Lambda')(-2\Lambda+2\Lambda') \quad . \quad 1 \quad . \quad 1 \\
-c_0c_{12}\vartheta_3\vartheta_{15} & - \quad 1 \quad . \quad 1 \quad . \quad 2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad -2\Lambda\cos\tfrac{1}{2}\pi(u+v)+2\Lambda'\cos\tfrac{1}{2}\pi(u-v) \\
+c_4c_8\vartheta_7\vartheta_{11} & + \quad 1 \quad . \quad 1 \quad . \ -2\Lambda\sin\tfrac{1}{2}\pi(u+v)-2\Lambda'\sin\tfrac{1}{2}\pi(u-v). \\
 & \qquad\qquad\qquad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v) \\
=0, & =0,
\end{array}$$

viz., the equation to be verified is here

$$\begin{array}{l}
-4\Lambda^2 \qquad\qquad\qquad +4\Lambda'^2 \\
+4\Lambda^2\cos^2\tfrac{1}{2}\pi(u+v)-4\Lambda'^2\cos^2\tfrac{1}{2}\pi(u-v) \\
+4\Lambda'^2\sin^2\tfrac{1}{2}\pi(u+v)-4\Lambda'^2\sin^2\tfrac{1}{2}\pi(u-v) \\
=0,\ \text{which is right.}
\end{array}$$

119. In the equation

$$\begin{array}{l|l}
c_9c_{12}\vartheta_1\vartheta_4\ \textit{i.e.} & 2Q.1.2Q\cos\tfrac{1}{2}\pi u.1 \\
-c_1c_4\vartheta_9\vartheta_{12} & -2Q.1.2Q\cos\tfrac{1}{2}\pi u.1 \\
+c_3c_6\vartheta_{14}\vartheta_{11} & \\
=0, & =0;
\end{array}$$

this is right, but there is no verification as to the term $c_3c_6\vartheta_{14}\vartheta_{11}$; taking the more approximate values, the term in question taken negatively, that is $-c_3c_6\vartheta_{14}\vartheta_{11}$ is =

$$-(2\Lambda+2\Lambda').\quad 2S.\quad -2S\sin\tfrac{1}{2}\pi v.\quad -2\Lambda\sin\tfrac{1}{2}\pi(u+v)+2\Lambda'\sin\tfrac{1}{2}\pi(u-v),$$

which is =

$$-8S^2(\Lambda+\Lambda')^2\cos\tfrac{1}{2}\pi u+8S^2(\Lambda+\Lambda')\Lambda\cos\tfrac{1}{2}\pi(u+2v)+8S^2(\Lambda+\Lambda')\Lambda'\cos\tfrac{1}{2}\pi(u-2v)$$

and this ought therefore to be the value of the first two terms, that is of

$$(2Q+2Q^9-2A-2A')(1-2Q^4-2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u$$
$$+2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u+2S^4\cos\pi v)$$
$$-(2Q+2Q^9+2A+2A')(1-2Q^4+2S^4)\{2Q\cos\tfrac{1}{2}\pi u+2Q^9\cos\tfrac{3}{2}\pi u$$
$$-2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}(1-2Q^4\cos\pi u-2S^4\cos\pi v),$$

which to the proper degree of approximation is

$$=(2Q-4Q^5-4QS^4+2Q^9-2A-2A')\{2Q\cos\tfrac{1}{2}\pi u-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u$$
$$+4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u+2A\cos\tfrac{1}{2}\pi(u+2v)+2A'\cos\tfrac{1}{2}\pi(u-2v)\}$$
$$-(2Q-4Q^5+4QS^4+2Q^9+2A+2A')\{2Q\cos\tfrac{1}{2}\pi v-4Q^5\cos\tfrac{1}{2}\pi u\cos\pi u$$
$$-4QS^4\cos\tfrac{1}{2}\pi u\cos\pi v+2Q^9\cos\tfrac{3}{2}\pi u-2A\cos\tfrac{1}{2}\pi(u+2v)-2A'\cos\tfrac{1}{2}\pi(u-2v)\}.$$

This is

$$(2M_0-2\Omega_0)(2M+2\Omega)$$
$$-(2M_0+2\Omega_0)(2M-2\Omega),\ =8(M_0\Omega-M\Omega_0)$$

if for a moment

$$M=Q\cos\tfrac{1}{2}\pi u-2Q^5\cos\tfrac{1}{2}\pi u\cos\pi u+Q^9\cos\tfrac{3}{2}\pi u,\qquad M_0=Q-2Q^5+Q^9,$$
$$\Omega=2QS^4\cos\pi u\cos\pi v+A\cos\tfrac{1}{2}\pi(u+2v)+A'\cos\tfrac{1}{2}\pi(u-2v),\ \Omega_0=2QS^4\quad +A+A',$$

or substituting and reducing, the value of $8(M_0\Omega-M\Omega_0)$ to the proper degree of approximation is found to be

$$=-8Q(2QS^4+A+A')\cos\tfrac{1}{2}\pi u$$
$$+8(Q^2S^4+8QA)\cos\tfrac{1}{2}\pi(u+2v)+8(Q^2S^4+8QA')\cos\tfrac{1}{2}\pi(u-2v),$$

which in virtue of the relations $QA=A^2S^2$, $QA'=A'^2S^2$, $Q^2S^2=AA'$, is equal to the foregoing value of $c_3c_6\vartheta_{14}\vartheta_{11}$. I have thought it worth while to give this somewhat elaborate verification.

Résumé of the foregoing results

120. In what precedes we have all the quadric relations between the 16 double theta-functions or say we have the linear relations between squares (squared functions) and the linear relations between pairs (products of two functions). the number of the asyzygetic linear relations between squares is obviously $=12$; and that of the asyzygetic linear relations between pairs is $=60$ (since each of the 30 tetrads of pairs gives two asyzygetic relations)· there are thus in all $12+60$, $=72$ asyzygetic linear relations. But these constitute only a 13-fold relation between the functions, viz.,

they are such as to give for the ratios of the 16 functions expressions depending upon two arbitrary parameters, x, y. Or taking the 16 functions as the coordinates of a point in 15-dimensional space, these coordinates are connected by a 13-fold relation (expressed by means of the foregoing system of 72 quadric equations), and the locus is thus a 13-fold, or two-dimensional, locus in 15-dimensional space.

Hence, taking any four of the functions, these are connected by a single equation;. that is regarding the four functions as the coordinates of a point in ordinary space, the locus of the point is a surface.

In particular the four functions may be any four functions belonging to a hexad. by what precedes there is then a linear relation between the squares of the four functions: or the locus is a quadric surface. Each hexad gives 15 such surfaces, or the number of quadric surfaces is $(16\times 15=)$ 240.

The 16-nodal quartic surfaces.

121. If the four functions are those contained in any two pairs out of a tetrad of pairs (see the foregoing "Table of the 120 pairs"), then the locus is a quartic surface, which is, in fact, a KUMMER's 16-nodal quartic surface. For if for a moment $x.y$ and $z.w$ are two pairs out of a tetrad, and $r.s$ be either of the remaining pairs of the tetrad; then we have rs a linear function of xy and zw: squaring, r^2s^2 is a linear function of x^2y^2, $xyzw$, z^2w^2; but we then have r^2 and s^2, each of them a linear function of x^2, y^2, z^2, w^2; or substituting we have an equation of the fourth order, containing terms of the second order in (x^2, y^2, z^2, w^2), and also a term in $xyzw$. It is clear that if instead of $r.s$ we had taken the remaining pair of the tetrad we should have obtained the same quartic equation in (x, y, z, w). And moreover it appears by inspection that if xy and zw are pairs in a tetrad, then xz and yw are pairs in a second tetrad, and xw and yz are pairs in a third tetrad: we obtain in each case the same quartic equation. We have from each tetrad of pairs six sets of four functions (x, y, z, w) and the number of such sets is thus $(\frac{1}{3}6.30=)60$: these are shown in the foregoing "Table of the 60 GOPEL tetrads," viz., taking as coordinates of a point the four functions in any tetrad of this table, the locus is a 16-nodal quartic surface.

122. To exhibit the process I take a tetrad 4, 7, 8, 11 containing two odd functions; and representing these for convenience by x, y, z, w, viz.: writing

$$\vartheta_4, \vartheta_7, \vartheta_8, \vartheta_{11}(u)=x, y, z, w$$

we have then X, Y, Z, W linear functions of the four squares, viz., it is easy to obtain

$$\begin{aligned}
\alpha(x^2+z^2)-\delta(y^2+w^2)&=2(\alpha^2-\delta^2)\mathrm{X},\\
\delta(\quad,,\quad)-\alpha(\quad,,\quad)&=2(\quad,,\quad)\mathrm{W},\\
-\beta(x^2-z^2)+\gamma(y^2-w^2)&=2(\beta^2-\gamma^2)\mathrm{Y},\\
-\gamma(\quad,,\quad)+\beta(\quad,,\quad)&=2(\quad,,\quad)\mathrm{Z}.
\end{aligned}$$

Also considering two other functions $\vartheta_0(u)$ and $\vartheta_{12}(u)$, or as for shortness I write them, ϑ_0 and ϑ_{12}, we have

$$\vartheta_0^2=\alpha X+\beta Y+\gamma Z+\delta W,$$
$$\vartheta_{12}^2=\alpha X-\beta Y-\gamma Z+\delta W,$$

and substituting the foregoing values of X, Y, Z, W, we find

$$M\vartheta_0^2=Ax^2+By^2+Cz^2+Dw^2,$$
$$M\vartheta_{12}^2=Cx^2+Dy^2+Az^2+Bw^2,$$

where writing down the values first in terms of α, β, γ, δ and then in terms of the c's, we have

$$\begin{aligned}
M&=(\alpha^2-\delta^2)(\beta^2-\gamma^2) &&=\tfrac{1}{4}\ .c_8^4-c_4^4,\\
A&=\beta^2\delta^2-\alpha^2\gamma^2 &&=\text{,,}\ \ -c_2^2c_6^2,\\
B&=-\alpha\delta(\beta^2-\gamma^2)+\beta\gamma(\alpha^2-\delta^2) &&=\text{,,}\ \ c_3^2c_4^2-c_{15}^2c_8^2,\\
C&=\alpha^2\beta^2-\gamma^2\delta^2 &&=\text{,,}\ \ c_1^2c_9^2,\\
D&=-\alpha\delta(\beta^2-\gamma^2)-\beta\gamma(\alpha^2-\delta^2) &&=\text{,,}\ \ c_{15}^2c_4^2-c_3^2c_8^2;
\end{aligned}$$

and we then have further

$$c_4c_8\vartheta_0\vartheta_{12}=c_0c_{12}\vartheta_4\vartheta_8+c_3c_{10}\vartheta_7\vartheta_{11},$$

that is

$$c_4c_8\vartheta_0\vartheta_{12}=c_0c_{12}xz\quad+c_3c_{10}yw;$$

whence equating the two values of $\vartheta_0^2\vartheta_{12}^2$ we have the required quartic equation in x, y, z, w.

123. But the reduction is effected more simply if instead of the c's we introduce the rectangular coefficients a, b, c, &c. We then have

$$M=(c''^2-b'^2),\quad A=-a''c,\quad C=a'b,$$
$$B=-b'c'-b'c'',=bc;\ D=b'b''+c'c'',=a'a'',$$

and the equations become

$$\begin{aligned}
(c''^2-b'^2)\vartheta_0^2&=-a''cx^2\quad+bcy^2+a'bz^2-a'a''w^2,\\
(c''^2-b'^2)\vartheta_{12}^2&=\ a'bx^2-a'a''y^2-a''cz^2+\ bcw^2\\
\sqrt{b'c''}\,\vartheta_0\vartheta_{12}&=\ \sqrt{a}\ xz+\sqrt{-b''c'}\,yw,
\end{aligned}$$

so that the elimination gives

$$b'c''(-a''cx^2+bcy^2+a'bz^2-a'a''w^2)$$
$$\times(\quad a'bx^2-a'a''y^2-a''cz^2+bcw^2)=(c''^2-b'^2)^2.\{ax^2z^2-b''c'y^2w^2+2\sqrt{-ab''c'}\,xyzw\},$$

viz.: this is

$$\begin{aligned}
&-a'a''bb'cc''(x^4+y^4+z^4+w^4)\\
&+a'b'cc''(a''^2+b^2)(x^2y^2+z^2w^2)\\
&+\{b'c''(a'^2b^2+a''^2c^2)-\quad a(b'^2-c''^2)^2\}x^2z^2\\
&+\{b'c''(a'^2a''^2+b^2c^2)+b''c'(b'^2-c''^2)^2\}y^2w^2\\
&-a''bb'c''(a'^2+c^2)(x^2w^2+y^2z^2)\\
&-2(b'^2-c''^2)^2\sqrt{-ab''c'}\,xyzw=0.
\end{aligned}$$

124. In this equation the coefficients of x^2z^2 and y^2w^2 are each $=a'a''bc(b'^2+c''^2)$, as at once appears from the identities

$$\begin{cases} a'b.b'-c''.a''c=a(b'^2-c''^2),\\ a'b.c''-b'.a''c=\ \ (b'^2-c'')^2,\end{cases}$$

$$\begin{cases} a'a''.b'-c''.bc=-b''(b'^2-c''^2),\\ a'a''.c-b'.bc=\quad c'(b'^2-c''^2),\end{cases}$$

by multiplying together in each pair the left hand and the right hand sides respectively. Substituting and dividing by $-a'a''bb'cc''$, we have

$$\begin{aligned}
&x^4+y^4+z^4+w^4\\
&-\frac{a''^2+b^2}{a''b}(x^2y^2+z^2w^2)-\frac{b'^2+c''^2}{b'c''}(x^2z^2+y^2w^2)+\frac{a'^2+c^2}{a'c}(x^2w^2+y^2z^2)\\
&+\frac{2(b'^2-c''^2)^2\sqrt{-ab''c'}}{a'a''bb'cc''}xyzw=0\ ;
\end{aligned}$$

or if we herein restore the c's in place of the rectangular coefficients this is

$$\begin{aligned}
&x^4+y^4+z^4+w^4\\
&-\frac{c_1^4+c_2^4}{c_1^2c_2^2}(x^2y^2+z^2w^2)-\frac{c_4^4+c_8^4}{c_4^2c_8^2}(x^2z^2+y^2w^2)+\frac{c_6^4+c_9^4}{c_6^2c_9^2}(x^2w^2+y^2z^2)\\
&+2\frac{c_0c_3c_{12}c_{15}(c_4^4-c_8^4)^2}{c_1^2c_2^2c_4^2c_6^2c_8^2c_9^2}xyzw=0,
\end{aligned}$$

which is the equation of the 16-nodal quartic surface.

Substituting for x, y, z, w their values ϑ_4, ϑ_7, ϑ_8, $\vartheta_{11}(u)$, we have the equation con-

necting the four theta-functions 4, 7, 8, 11 of a GOPEL tetrad. And there is an equation of the like form between the four functions of any other GOPEL tetrad: for obtaining the actual equations some further investigation would be necessary.

The xy-expressions of the theta-functions.

125. The various quadric relations between the theta-functions, admitting that they constitute a 13-fold relation, show that the theta-functions may be expressed as proportional to functions of two arbitrary parameters x, y; and two of these functions being assumed at pleasure the others of them would be determinate; we have of course (though it would not be easy to arrive at it in this manner) such a system in the foregoing expressions of the 16 functions in terms of x, y; and conversely these expressions must satisfy identically the quadric relations between the theta-functions.

126. To show that this is so as to the general form of the equations, consider first the xy-factors $\sqrt{a}$, $\sqrt{ab}$, &c. As regards the squared functions $(\sqrt{ab})^2$, we have for instance

$$(\sqrt{ab})^2=\frac{1}{\theta^2}\{\mathrm{abfc_{\prime}d_{\prime}e_{\prime}+a_{\prime}b_{\prime}f_{\prime}cde}+2\sqrt{\mathrm{XY}}\},$$

$$(\sqrt{cd})^2=\frac{1}{\theta^2}\{\mathrm{cdfa_{\prime}b_{\prime}e_{\prime}+c_{\prime}d_{\prime}f_{\prime}abe}+2\sqrt{\mathrm{XY}}\},$$

each of these contains the same irrational part $\frac{2}{\theta^2}\sqrt{\mathrm{XY}}$, and the difference is therefore rational; and it is moreover integral, for we have

$$(\sqrt{ab})^2-(\sqrt{cd})^2=\frac{1}{\theta^2}(\mathrm{abc_{\prime}d_{\prime}-a_{\prime}b_{\prime}cd})(\mathrm{fe_{\prime}-f_{\prime}e}),$$

where each factor divides by θ, and consequently the product by θ^2; the value is in fact

$$=(e-f)\begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix},$$

a linear function of $1, x+y, xy$; and this is the case as regards the difference of any two of the squares $(\sqrt{ab})^2$, $(\sqrt{ac})^2$, &c.; hence selecting any one of these squares for instance $(\sqrt{de})^2$, any other of the squares is of the form

$$\lambda+\mu(x+y)+\nu xy+\rho(\sqrt{de})^2;\ (\rho=1)$$

and obviously, the other squares $(\sqrt{a})^2$, &c., are of the like form, the last coefficient ρ

being $=0$. We hence have the theorem that each square can be expressed as a linear function of any four (properly selected) squares.

127. But we have also the theorem of the 16 KUMMER hexads.

Obviously the six squares

$$(\sqrt{a})^2,\ (\sqrt{b})^2,\ (\sqrt{c})^2,\ (\sqrt{d})^2,\ (\sqrt{e})^2,\ (\sqrt{f})^2$$

are a hexad, viz.: each of these is a linear function of 1, $x+y$, xy, and therefore selecting any three of them, each of the remaining three can be expressed as a linear function of these.

But further the squares $(\sqrt{a})^2$, $(\sqrt{b})^2$, $(\sqrt{ab})^2$, $(\sqrt{cd})^2$, $(\sqrt{ce})^2$, $(\sqrt{de})^2$ form a hexad. For reverting to the expression obtained for $(\sqrt{ab})^2-(\sqrt{cd})^2$, the determinant contained therein is a linear function of $\mathrm{aa}_{,}$ and $\mathrm{bb}_{,}$, that is of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; we in fact have

$$(a-b)\begin{vmatrix} 1, & x+y, & xy \\ 1, & a+b, & ab \\ 1, & c+d, & cd \end{vmatrix}=(b-c)(b-d)(a-x)(a-y)-(a-c)(a-d)(b-x)(b-y)$$

Hence $(\sqrt{ab})^2-(\sqrt{cd})^2$ is a linear function of $(\sqrt{a})^2$, $(\sqrt{b})^2$; and by a mere interchange of letters $(\sqrt{ab})^2-(\sqrt{ce})^2$, $(\sqrt{ab})^2-(\sqrt{de})^2$, are each of them also a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$; whence the theorem. And we have thus all the remaining 15 hexads.

128. We have a like theory as regards the products of pairs of functions; a tetrad of pairs is of one of the two forms

$$\sqrt{a}\sqrt{b},\ \sqrt{ac}\sqrt{bc},\ \sqrt{ad}\sqrt{bd},\ \sqrt{ae}\sqrt{be} \text{ and } \sqrt{f}\sqrt{ab},\ \sqrt{c}\sqrt{de},\ \sqrt{d}\sqrt{ce},\ \sqrt{e}\sqrt{cd};$$

in the first case the terms are

$$\sqrt{\mathrm{aa_{,}bb_{,}}}\},$$

$$\frac{1}{\theta^2}\{(\mathrm{ab_{,}+a_{,}b})\sqrt{\mathrm{cdefc_{,}d_{,}e_{,}f_{,}}}+(\mathrm{cfd_{,}e_{,}+c_{,}f_{,}de})\sqrt{\mathrm{aa_{,}bb_{,}}}\},$$

$$\frac{1}{\theta^2}\{\quad ,, \qquad\qquad ,, \qquad +(\mathrm{dfc_{,}e_{,}+d_{,}f_{,}ce}) \quad ,, \quad \},$$

$$\frac{1}{\theta^2}\{\quad ,, \qquad\qquad ,, \qquad +(\mathrm{efc_{,}d_{,}+c_{,}f_{,}ce}) \quad ,, \quad \},$$

and as regards the last three terms the difference of any two of them is a mere constant multiple of $\sqrt{\mathrm{aa_{,}bb_{,}}}$; for instance, the second term $-$ the third term is

$=\frac{1}{\theta^2}(\mathrm{cd}_,-\mathrm{c}_,\mathrm{d})(\mathrm{fe}_,-\mathrm{f}_,\mathrm{e})\sqrt{\overline{\mathrm{aa}_,\mathrm{bb}_,}}_{,,}=(c-d)(f-e)\sqrt{\mathrm{aa}_,\overline{\mathrm{bb}}_,}$; we have thus a tetrad such that selecting any two terms, each of the remaining terms is a linear function of these.

In the second case the terms are

$$\begin{array}{llll} \frac{1}{\theta}\{\mathrm{f}\sqrt{\mathrm{abc}_,\overline{\mathrm{d}_,\mathrm{e}_,\mathrm{f}_,}} & + & \mathrm{f}_,\sqrt{\mathrm{a}_,\mathrm{b}_,\mathrm{cdef}}\}, \\ \frac{1}{\theta}\{\mathrm{c} \quad ,, & + & \mathrm{c}_, \quad ,, & \}, \\ \frac{1}{\theta}\{\mathrm{d} \quad ,, & + & \mathrm{d}_, \quad ,, & \}, \\ \frac{1}{\theta}\{\mathrm{e} \quad ,, & + & \mathrm{e}_, \quad ,, & \}, \end{array}$$

whence clearly the four terms are a tetrad as above. And it may be added that any linear function of the four terms is of the form

$$\frac{1}{\theta}\{(\lambda+\mu x)\sqrt{\overline{\mathrm{abc}_,\mathrm{d}_,\mathrm{e}_,\mathrm{f}_,}}+(\lambda+\mu y)\sqrt{\mathrm{a}_,\mathrm{b}_,\overline{\mathrm{cdef}}}\}.$$

129. Considering next the actual equations between the squared theta-functions, take as a specimen

$$c_6^2 \quad \vartheta_6^2-c_2^2 \quad \vartheta_2^2+c_1^2 \quad \vartheta_1^2-c_9^2 \quad \vartheta_9^2=0,$$

that is

$$c_6^4(\sqrt{ab})^2-c_2^4(\sqrt{cd})^2+c_1^4(\sqrt{ce})^2-c_9^4(\sqrt{de})^2=0,$$

where $c_6, c_2, c_1, c_9=\sqrt[4]{ab}, \sqrt[4]{cd}, \sqrt[4]{ce}, \sqrt[4]{de}$ respectively. Since the functions $(\sqrt{ab})^2$, &c., contain the same irrational term $\frac{2}{\theta^2}\sqrt{XY}$, it is clear that the equation can only be true if

$$c_6^4-c_2^4+c_1^4-c_9^4=0,$$

and this being so it will be true if

$$c_2^4\{(\sqrt{ab})^2-(\sqrt{cd})^2\}-c_1^4\{(\sqrt{ab})^2-(\sqrt{ce})^2\}+c_9^4\{(\sqrt{ab})^2-(\sqrt{de})^2\}=0,$$

where by what precedes each of the terms in $\{\}$ is a linear function of $(\sqrt{a})^2$ and $(\sqrt{b})^2$: attending first to the term in $(\sqrt{a})^2$, the coefficient hereof is

$$ef.bc\ bd.c_2^4-df.bc.be.c_1^4+cf.bd.be.c_9^4,$$

where for shortness bc, bd, &c, are written to denote the differences $b-c$, $b-d$, &c : substituting for c_2^4 its value $(\sqrt{cd})^4$, $=cd.cf.df.ab\ ae.be$, and similarly for c_1^4 and c_9^4 their values, $=ce.cf.ef.ab.ad.bd$, and $de.df.ef.ab.ac.bc$ respectively, the whole expression

contains the factor $ab.bc.bd.be.cf.df.ef$, and throwing this out, the equation to be verified becomes

$$cd.ae-ce.ad+de.ac=0$$

which is true identically. The verification is thus made to depend upon that of $c_6^4-c_2^4+c_1^4-c_9^4=0$; and similarly for the other relations between the squared functions, the verification depends upon relations containing the fourth powers, or the products of squares, of the constants c and k.

130. Among these are included the before-mentioned system of equations involving the fourth powers or the products of squares of only the constants c; and it is interesting to show how these are satisfied identically by the values $c_0=\sqrt[4]{bd}$, &c.

Thus one of these equations is $c_{12}^4+c_1^4+c_6^4=c_0^4$; substituting the values, there is a factor ce which divides out, and the resulting equation is

$$ad.af.df.bc.be+cf.ef.ab.ad.bd+ab\,af.bf.cd.de-ac.ae.bd.bf.df=0.$$

There are here terms in a^2, a, a^0 which should separately vanish; for the terms in a^2 the equation becomes

$$df.bc.be+bd.cf.ef+bf\,cd.de-bd.bf.df=0,$$

which is easily verified, and the equations in a and a^0 may also be verified.

An equation involving products of the squares is $c_{12}^2c_0^2-c_1^2c_4^2+c_3^2c_6^2=0$. The term $c_{12}^2c_0^2$ is here $\sqrt{adf.bce}\,\sqrt{def.abc}$ which is $=\sqrt{(bc)^2(df)^2.ab\,ac.ad.af\,be.ce.de.ef}$, which is taken $=bc.df\sqrt{ab.ac.ad.af.be.ce.de.ef}$; similarly the values of $c_1^2c_4^2$ and $c_3^2c_6^2$ are $=bd.cf$ and $bf.cd$ each into the same radical, and the equation to be verified is

$$bc.df-bd.cf+bf.cd=0,$$

which is right: and the other equations may be verified in a similar manner.

131. Coming next to the equations connecting the pairs of theta-functions, for instance

$$c_3c_{15}\vartheta_0\vartheta_{12}-c_0c_{12}\vartheta_3\vartheta_{15}+c_1c_5\vartheta_7\vartheta_{11}=0,$$

this is

$$c_3c_{15}c_0c_{12}\{\sqrt{bd}\,\sqrt{ad}-\sqrt{be}\sqrt{ae}\}+c_1c_5k_7k_{11}\,\sqrt{b}\sqrt{a}=0,$$

the products $\sqrt{bd}\,\sqrt{ad}$ and $\sqrt{be}\,\sqrt{ae}$ contain besides a common term the terms $\frac{1}{\theta^2}(\mathrm{dfc_,e_,+d_,f_,ce})\sqrt{\mathrm{aa_,bb_,}}$, and $\frac{1}{\theta^2}(\mathrm{efc_,d_,+e_,f_,cd})\sqrt{\mathrm{aa_,bb_,}}$, hence their difference contains $\frac{1}{\theta^2}(\mathrm{de_,-d_,e})(\mathrm{fc_,-f_,c})\sqrt{\mathrm{aa_,bb_,}}$ which is $=de.fc\sqrt{\mathrm{aa_,bb_,}}$, that is $de.fc\sqrt{a}\sqrt{b}$: hence the equation to be verified is

$$de.fc.c_3c_{15}c_0c_{12}+c_4c_5k_7k_{11}=0;$$

$c_3c_{15}c_0c_{12}$ is $=\sqrt[4]{bef.acd}\;\sqrt[4]{aef.bcd}\;\sqrt[4]{bdf.ace}\;\sqrt[4]{adf\,bce}$, where under the fourth root we have 24 factors, which are, in fact, 12 factors twice repeated; and if we write

$\Pi, = ab.ac.ad.ae.af.bc.bd.be.bf.cd.ce.cf.de.df.ef$, for the product of all the 15 factors, then the 12 factors are in fact all those of Π, except ab, cf, de; viz., we have

$$c_3 c_{15} c_0 c_{12} = \sqrt[4]{\Pi^2 \div (ab)^2 (cf)^2 (de)^2}.$$

Again, $c_4 c_8 k_7 k_{11}, = \sqrt[4]{acf.bde}\ \sqrt[4]{bcf.ade}\ \sqrt[4]{acdef}\ \sqrt[4]{bcdef}$, is a fourth root of a product of 32 factors, which are in fact 16 factors twice repeated, and in the 16 factors, ab does not occur, cf and de occur each twice, and the other 12 factors each once: we thus have

$$c_4 c_8 k_7 k_{11} = \sqrt[4]{\Pi^2 (cf)^2 (de)^2 \div (ab)^2},$$

and the relation to be verified assumes the form

$$fc.de \sqrt[4]{1 \div (cf)^2 (de)^2} + \sqrt[4]{(cf)^2 (de)^2} = 0,$$

which, taking $fc.de = -\sqrt[4]{(cf)^4 (de)^4}$, is right. And so for the other equations. It will be observed that in the equation $de.fc.c_3 c_{15} c_0 c_{12} + c_4 c_8 k_7 k_{11} = 0$, and the other equations upon which the verifications depend, there is no ambiguity of sign: the signs of the radicals have to be determined consistently with all the equations which connect the c's and the k's: that this is possible appears evident *à priori*, but the actual verification presents some difficulty. I do not here enter further into the question.

Further results of the product-theorem, the $u \pm u'$ formulæ.

132. Recurring now to the equations in $u+u'$, $u-u'$, by putting therein $u'=0$, we can express X, Y, Z, W in terms of four of the squared functions of u, and by putting $u=0$ we can express X′, Y′, Z′, W′ in terms of four of the squared functions of u'; and, substituting in the original equations, we have the products $\vartheta(\ )u+u'.\vartheta(\ )u-u'$ in terms of the squared functions of u and u'.

Selecting as in a former investigation the functions 4, 7, 8, 11 (which were called x, y, z, w) it is more convenient to use single letters for representing the squared functions, and I write

$\vartheta(u+u')\ .\ \vartheta(u-u')$				$\vartheta^2 u$			$\vartheta^2 u'$			$\vartheta^2 0$		
4	4	=	P,	4	=	p,	4	=	p',	4	=	$p_0\ (=c_4^2)$,
7	7	=	Q,	7	=	q,	7	=	q',	7	=	0,
8	8	=	R,	8	=	r,	8	=	r',	8	=	$r_0\ (=c_8^2)$,
11	11	=	S,	11	=	s,	11	=	s',	11	=	0.

Then

$$\begin{array}{llllllllllllllll}
 & X & Y & Z & W & & X & Y & Z & W & & X' & Y' & Z' & W' \\
P = & X' & -Y' & +Z' & -W', & p= & \alpha & -\beta & +\gamma & -\delta, & p'= & \alpha & -\beta & +\gamma & -\delta, \\
Q = & W' & -Z' & +Y' & -X', & q= & \delta & -\gamma & +\beta & -\alpha, & q'= & \delta & -\gamma & +\beta & -\alpha, \\
R = & X' & +Y' & -Z' & -W', & r= & \alpha & +\beta & -\gamma & -\delta, & r'= & \alpha & +\beta & -\gamma & -\delta, \\
S = & W' & +Z' & -Y' & -X', & s= & \delta & +\gamma & -\beta & -\alpha, & s'= & \delta & +\gamma & -\beta & -\alpha.
\end{array}$$

Hence

$$\begin{array}{ll}
\alpha(p+r)-\delta(q+s)=2(\alpha^2-\delta^2)X, & \alpha(p'+r')-\delta(q'+s')=2(\alpha^2-\delta^2)X', \\
\delta \quad ,, \quad -\alpha \quad ,, \quad =2 \quad ,, \quad W, & \delta \quad ,, \quad -\alpha \quad ,, \quad =2 \quad ,, \quad W, \\
-\beta(p-r)+\gamma(q-s)=2(\beta^2-\gamma^2)Y, & -\beta(p'-r')+\gamma(q'-s')=2(\beta^2-\gamma^2)Y', \\
-\gamma \quad ,, \quad +\beta \quad ,, \quad =2 \quad ,, \quad Z, & -\gamma \quad ,, \quad +\beta \quad ,, \quad =2 \quad ,, \quad Z'.
\end{array}$$

and by means of these values

$$\begin{array}{l}
4(\alpha^2-\delta^2)^2X'X=\alpha^2(p+r)(p'+r')+\delta^2(q+s)(q'+s')-\alpha\delta[(p+r)(q'+s')+(p'+r')(q+s)], \\
4 \quad ,, \quad W'W=\delta^2 \quad ,, \quad +\alpha^2 \quad ,, \quad -\alpha\delta[\quad ,, \quad ,, \quad] \\
4(\beta^2-\gamma^2)^2Y'Y=\beta^2(p-r)(p'-r')+\gamma^2(q-s)(q'-s')-\beta\gamma[(p-r)(q'-s')+(p'-r')(q-s)], \\
4 \quad ,, \quad Z'Z=\gamma^2 \quad ,, \quad +\beta^2 \quad ,, \quad -\beta\gamma[\quad ,, \quad ,, \quad]
\end{array}$$

Hence

$$4(\alpha^2-\delta^2)(X'X-W'W)=(p+r)(p'+r')-(q+s)(q'+s'),$$
$$4(\beta^2-\gamma^2)(Y'Y-Z'Z)=(p-r)(p'-r')-(q-s)(q'-s'),$$

and substituting in the expressions for P and R,

$$\begin{array}{l}
4(\alpha^2-\delta^2)(\beta^2-\gamma^2)P= \\
\quad (\beta^2-\gamma^2)[(p+r)(p'+r')-(q+s)(q'+s')]-(\alpha^2-\delta^2)[(p-r)(p'-r')-(q-s)(q'-s')], \\
4 \quad ,, \quad R= \\
\quad ,, \; [\quad ,, \quad ,, \quad]+ \; ,, \; [\quad ,, \quad ,, \quad]
\end{array}$$

Similarly

$$4(\alpha^2-\delta^2)^2\mathrm{W'X} =$$
$$\alpha\delta[(p+r)(p'+r')+(q+s)(q'+s')]-\alpha^2(p+r)(q'+s')-\delta^2(q+s)(p'+r'),$$
$$4 \quad \text{,,} \quad \mathrm{X'W}=$$
$$\text{,,}\,[\quad \text{,,} \qquad \text{,,} \quad]-\delta^2 \quad \text{,,} \qquad -\alpha^2 \quad \text{,,} \quad ,$$
$$4(\beta^2-\gamma^2)^2\mathrm{Z'Y} =$$
$$\beta\gamma[(p-r)(p'-r')+(q-s)(q'-s')]-\beta^2(p-r)(q'-s')-\gamma^2(q-s)(p'-r'),$$
$$4 \quad \text{,,} \quad \mathrm{Y'Z} =$$
$$\text{,,}\,[\quad \text{,,} \qquad \text{,,} \quad]-\gamma^2 \quad \text{,,} \qquad -\beta^2 \quad \text{,,} \quad ,$$

whence

$$4(\alpha^2-\delta^2)(\mathrm{W'X}-\mathrm{X'W})=-[(p+r)(q'+s')-(p'+r')(q+s)],$$
$$4(\beta^2-\gamma^2)(\mathrm{Z'Y}-\mathrm{Y'Z})=-[(p-r)(q'-s')-(p'-r')(q-s)],$$

and substituting in the expressions for Q and S

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)\mathrm{Q}=$$
$$-(\beta^2-\gamma^2)[(p+r)(q'+s')-(p'+r')(q+s)]+(\alpha^2-\delta^2)[(p-r)(q'-s')-(p'-r')(q-s)],$$
$$4 \quad \text{,,} \quad \mathrm{R}=$$
$$- \quad \text{,,} \quad [\quad \text{,,} \qquad \text{,,} \quad]- \quad \text{,,} \quad [\quad \text{,,} \qquad \text{,,} \quad]$$

133. Hence collecting and reducing

$$4(\alpha^2-\delta^2)(\beta^2-\gamma^2)\mathrm{P}=$$
$$-(\alpha^2-\beta^2+\gamma^2-\delta^2)(pp'-qq'+rr'-ss')+(\alpha^2+\beta^2-\gamma^2-\delta^2)(pr'+p'r-qs'-q's),$$
$$4 \quad \text{,,} \quad \mathrm{R}=$$
$$(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad)-(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad),$$
$$4 \quad \text{,,} \quad \mathrm{Q}=$$
$$(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad)-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad),$$
$$4 \quad \text{,,} \quad \mathrm{S}=$$
$$-(\alpha^2+\beta^2-\gamma^2-\delta^2)(\quad \text{,,} \quad)+(\alpha^2-\beta^2+\gamma^2-\delta^2)(\quad \text{,,} \quad),$$

we have $p_0(=c_4^2)=\alpha^2-\beta^2+\gamma^2-\delta^2$, $r_0(=c_8^2)=\alpha^2+\beta^2-\gamma^2-\delta^2$, and thence

$$r_0^2-p_0^2=4(\alpha^2-\delta^2)(\beta^2-\gamma^2);$$

the equations hence become

$$
\begin{aligned}
(r_0^2-p_0^2)\mathrm{P} &= -p_0(pp'-qq'+rr'-ss')+r_0(pr'+p'r-qs'-q's),\\
\text{,,}\quad \mathrm{R} &= r_0(\qquad \text{,,} \qquad)-p_0(\qquad \text{,,} \qquad),\\
\text{,,}\quad \mathrm{Q} &= p_0(pq'-p'q+rs'-r's)-r_0(\qquad \text{,,} \qquad),\\
\text{,,}\quad \mathrm{S} &= -r_0(\qquad \text{,,} \qquad)+p_0(\qquad \text{,,} \qquad).
\end{aligned}
$$

On writing in the equations $u'=0$, then P, Q, R, S, p', q', r', s' become $=p, q, r, s, p_0, 0, r_0, 0$; and the equations are (as they should be) true identically. The equations may be written

$(c^4\ c^4)$	$\overset{u+u'}{\vartheta}$.	$\overset{u-u'}{\vartheta}$		$c^2(\vartheta''^2.\underline{\vartheta''^2}\ \ \underline{\vartheta''^2}.\vartheta''^2\ \ \vartheta^2.\underline{\vartheta'^2}\ \ \underline{\vartheta'^2}.\underline{\vartheta'^2})$	$c^2(\vartheta''^2.\vartheta^2\ \ \vartheta^2\vartheta''^2\ \ \vartheta'^2.\underline{\vartheta'^2}\ \ \underline{\vartheta''^2}.\vartheta'^2)$
(8−4)	4	4	=	−4(4.4 − 7.7 +8.8 −11.11)	+8(4 8 + 8.4−7.11−11.7),
(,,)	8	8	=	+8(,,)	−4(,,),
(,,)	7	7	=	+4(4.7 − 7.4 +8.11−11.8)	−8(4.11−11.4+8.7 − 7.8),
(,,)	11	11	=	−8(,,)	+4(,,),

and there is of course such a system for each of the 60 GÖPEL tetrads.

Differential relations connecting the theta-functions with the quotient-functions.

134. Imagine p, q, r, s, &c., changed into x^2, y^2, z^2, w^2; that is, let x, y, z, w represent the theta-functions 4, 7, 8, 11 of u, v; and similarly x', y', z', w' those of u', v', and x_0, 0, z_0, 0 those of 0, 0. Let u', v' be each of them indefinitely small; and take $\delta, =u'\frac{d}{du}+v'\frac{d}{dv}$, as the symbol of total differentiation in regard to u, v, the infinitesimals u' and v' being arbitrary: then we have in general

$$\vartheta(u+u', v+v)=\vartheta(u, v)+\delta\vartheta(u, v)+\tfrac{1}{2}\delta^2\vartheta(u, v),$$

and hence

$$\mathrm{P}=(x+\delta x+\tfrac{1}{2}\delta^2x)(x-\delta x+\tfrac{1}{2}\delta^2x),\ =x^2+(x\delta^2x-(\delta x)^2),$$

and similarly for Q, R, S. Moreover, observing that x', z' are even functions, y', z' odd functions of (u', v'), we have

$$x', y', z', w'=x_0+\tfrac{1}{2}\delta^2x_0,\ \delta y_0,\ z_0+\tfrac{1}{2}\delta^2z_0,\ \delta w_0,$$

where δ^2x_0, δy_0, &c. are what δ^2x, δy, &c., become on writing therein $u=0$, $v=0$; δy_0, δw_0 are of course linear functions, δ^2x_0, δ^2z_0 quadric functions of u' and v'. The values of x'^2, y'^2, z'^2, w'^2 are thus $x_0^2+x_0\delta^2x_0$, $(\delta y_0)^2$, $z_0^2+z_0\delta^2z_0$, $(\delta w_0)^2$; and we have

$$
\begin{array}{llllllllll}
 & & & & & & x_0\delta^2x_0 & (\delta y_0)^2 & z_0\delta^2z_0 & (\delta w_0)^2 \\
x^2x'^2-y^2y'^2+z^2z'^2-w^2w'^2 &=& x^2x_0^2 &+z^2z_0^2 & & & +x^2 & -y^2 & +z^2 & -w^2, \\
x^2y'^2-y^2x'^2+z^2w'^2-w^2z'^2 &=& -y^2x_0^2 &-w^2z_0^2 & & & -y^2 & +x^2 & -w^2 & +z^2, \\
x^2z'^2-y^2w'^2+z^2x'^2-w^2y'^2 &=& z^2x_0^2 &+x^2z_0^2 & & & +z^2 & -w^2 & +x^2 & -y^2, \\
x^2w'^2-y^2z'^2+z^2y'^2-w^2x'^2 &=& -w^2x_0^2 &-y^2z_0^2 & & & -w^2 & +z^2 & -y^2 & +x^2.
\end{array}
$$

135. On substituting these values the constant terms (or terms independent of u', v') disappear of themselves; and the equations (transposing the second and third of them) become

$$
\begin{array}{lllll}
 & \overline{\qquad x_0\delta^2x_0 \qquad} & \overline{\qquad (\delta y_0)^2 \qquad} & \overline{\qquad z_0\delta^2z_0 \qquad} & \overline{\qquad (\delta w_0)^2 \qquad} \\
(z_0^4-x_0^4)\{x\delta^2x-(\delta x)^2\}= & (-x_0^2x^2+z_0^2z^2) & +(\ x_0^2y^2-z_0^2w^2) & +(-x_0^2z^2+z_0^2x^2) & +(\ x_0^2w^2-z_0^2y^2), \\
\quad ,, \qquad \{y\delta^2y-(\delta y)^2\}= & -(\ x_0^2y^2-z_0^2w^2) & -(-x_0^2x^2+z_0^2z^2) & -(\ x_0^2w^2-z_0^2y^2) & -(-x_0^2z^2+z_0^2x^2), \\
\quad ,, \qquad \{z\delta^2z-(\delta z)^2\}= & (-x_0^2z^2+z_0^2x^2) & +(\ x_0^2w^2-z_0^2y^2) & +(-x_0^2x^2+z_0^2z^2) & +(\ x_0^2y^2-z_0^2w^2), \\
\quad ,, \qquad \{w\delta^2w-(\delta w)^2\}= & -(\ x_0^2w^2-z_0^2y^2) & -(-x_0^2z^2+z_0^2x^2) & -(\ x_0^2y^2-z_0^2w^2) & -(-x_0^2x^2+z_0^2z^2),
\end{array}
$$

where it will be recollected that x, y, z, w mean $\vartheta_4, \vartheta_7, \vartheta_8, \vartheta_{11}\,(u)$; x_0 is $\vartheta_4\,(0)$ that is c_4, and z_0 is $\vartheta_8\,(0)$ that is c_8. But the formulæ contain also

$$
\begin{aligned}
\delta^2x_0&=(c_4''', c_4'', c_4'\between u', v')^2, & \delta y_0&=(c_7', c_7''\between u', v'), \\
\delta^2z_0&=(c_8''', c_8'', c_8'\between u', v')^2, & \delta w_0&=(c_{11}', c_{11}''\between u', v').
\end{aligned}
$$

The formulæ may be written

$$
\begin{array}{llllll}
 & & \underbrace{\qquad c_4\delta^2c_4 \qquad} & \underbrace{\qquad (\delta c_7)^2 \qquad} & \underbrace{\qquad c_8\delta^2c_8 \qquad} & \underbrace{\qquad (\delta c_{11})^2 \qquad} \\
 & \{\ \vartheta \quad \delta^2\vartheta-(\delta\vartheta)^2\} & c^2\ \vartheta^2\ \ c^2\ \vartheta^2 & c^2\ \vartheta^2\ \ c^2\ \vartheta^2 & c^2\ \vartheta^2\ \ c^2\ \vartheta^2 & c^2\ \vartheta^2\ \ c^2\ \vartheta^2 \\
(c_8^4-c_4^4) & \{\ 4 \quad 4 \quad 4\ \} = & (-4\ \ 4\ +8\ \ 8) & +(\ 4\ \ 7\ -8\ 11) & +(-4\ \ 8\ +8\ \ 4) & +(\ 4\ 11\ -8\ \ 7), \\
\quad ,, & \{\ 7 \quad 7 \quad 7\ \} = & -(\ 4\ \ 7\ -8\ 11) & -(-4\ \ 4\ +8\ \ 8) & -(\ 4\ 11\ -8\ \ 7) & -(-4\ \ 8\ +8\ \ 4), \\
\quad ,, & \{\ 8 \quad 8 \quad 8\ \} = & (-4\ \ 8\ +8\ \ 4) & +(\ 4\ 11\ -8\ \ 7) & +(-4\ \ 4\ +8\ \ 8) & +(\ 4\ \ 7\ -8\ 11), \\
\quad ,, & \{11 \quad 11 \quad 11\} = & -(\ 4\ 11\ -8\ \ 7) & +(-4\ \ 8\ +8\ \ 4) & -(\ 4\ \ 7\ -8\ 11) & -(-4\ \ 4\ +8\ \ 8),
\end{array}
$$

where $\delta^2c_4, \delta^2c_8, \delta c_7, \delta c_{11}$ are written in place of $\delta^2x_0, \delta^2z_0, \delta y_0, \delta z_0$ There is of course a like system of equations for each of the GOPEL tetrads.

136. Observe that dividing the first equation by $\vartheta_4^2(u)$, or say by ϑ_4^2, the left hand side is a mere constant multiple of $\delta^2 \log \vartheta_4$, and the right hand side depends only on the quotient-functions $\vartheta_7\div\vartheta_4, \vartheta_8\div\vartheta_4, \vartheta_{11}\div\vartheta_4$; each side is a quadric function of (u', v'), and equating the terms in $u'^2, u'v', v'^2$ respectively, we have

$$
\frac{d^2}{du^2}\log\vartheta_4, \quad \frac{d^2}{du\,dv}\log\vartheta_4, \quad \frac{d^2}{dv^2}\log\vartheta_4
$$

each of them expressed as a linear function of the squares of the quotient-functions $\vartheta_7\div\vartheta_4, \vartheta_8\div\vartheta_4, \vartheta_{11}\div\vartheta_4$. The formula is thus a second-deviative formula serving for the expression of a double theta-function by means of three quotient-functions.

Differential relations of the theta-functions.

137. In "The second set of 16," selecting the eight equations which contain Y_1 and W_1, these are

$$\begin{array}{rrcrrcl}
 & \overset{u+u'}{\vartheta} \,. & \overset{u-u'}{\vartheta} & & \overset{u+u'}{\vartheta} \,. & \overset{u-u'}{\vartheta} & \text{(Suffixes 1)} \\
 & & & & & & \overbrace{\mathrm{Y} \quad\quad \mathrm{W}} \\
\tfrac{1}{2}\{ & 4 & 0 & - & 0 & 4\} & = \mathrm{Y}' + \mathrm{W}', \\
 & 12 & 8 & - & 8 & 12 & = \mathrm{Y}' - \mathrm{W}', \\
 & 6 & 2 & - & 2 & 6 & = \mathrm{W}' + \mathrm{Y}', \\
 & 14 & 10 & - & 10 & 14 & = \mathrm{W}' - \mathrm{Y}', \\
\hline
\tfrac{1}{2}\{ & 5 & 1 & + & 1 & 5\} & = \mathrm{X}' + \mathrm{Z}', \\
 & 13 & 9 & + & 9 & 13 & = \mathrm{X}' - \mathrm{Z}', \\
 & 7 & 3 & + & 3 & 7 & = \mathrm{Z}' + \mathrm{X}', \\
 & 15 & 11 & + & 11 & 15 & = \mathrm{Z}' - \mathrm{X}',
\end{array}$$

and then, considering any line in the upper half and any two lines in the lower half, we can from the three equations eliminate Y_1 and W_1, thus obtaining an equation such as

$$\begin{vmatrix} \vartheta_4 \vartheta_0 - \vartheta_0\vartheta_4, & \mathrm{Y}', & \mathrm{W}' \\ \vartheta_5 \vartheta_1 + \vartheta_1\vartheta_5, & \mathrm{X}', & \mathrm{Z}' \\ \vartheta_{13}\vartheta_9 + \vartheta_9\vartheta_{13}, & \mathrm{X}', & -\mathrm{Z}' \end{vmatrix} = 0,$$

viz., this is

$$\begin{aligned} &-2\mathrm{X}'\mathrm{Z}' \qquad\qquad (\vartheta_4 \vartheta_0 - \vartheta_0\vartheta_4) \\ &+(\ \ \mathrm{X}'\mathrm{W}' + \mathrm{Y}'\mathrm{Z}')(\vartheta_5 \vartheta_1 + \vartheta_1\vartheta_5) \\ &+(-\mathrm{X}'\mathrm{W}' + \mathrm{Y}'\mathrm{Z}')(\vartheta_{13}\vartheta_9 + \vartheta_9\vartheta_{13}) = 0, \end{aligned}$$

where the arguments of the theta-functions are as above, $u+u'$, $u-u'$, $u+u'$, $u-u'$; and the suffixes of the X', Y', Z', W' are all $=1$.

138. Suppose in this equation u' becomes indefinitely small; if u' were $=0$ the values of X', Y', Z', W' would be α, 0, γ, 0: and hence u' being indefinitely small we take them to be α, $\delta\beta$, γ, $\delta\delta$, where

$$\delta\beta, = \left(u'\frac{d}{du} + v'\frac{d}{dv}\right)\mathrm{Y}, \text{ and } \delta\delta, = \left(u'\frac{d}{du} + v'\frac{d}{dv}\right)\mathrm{W}, \qquad (u=v=0).$$

are in fact linear functions of u' and v'.

We have $\vartheta_4\vartheta_0-\vartheta_0\vartheta_4$ standing for

$$\vartheta_4(u+u')\vartheta_0(u-u')-\vartheta_0(u+u')\vartheta_4(u-u'),$$

and here $\vartheta_4(u\pm u')=\vartheta_4\pm\delta\vartheta_4$, $\vartheta_0(u\pm u')=\vartheta_0\pm\delta\vartheta_0$; the function in question is thus

$$(\vartheta_4+\delta\vartheta_4)(\vartheta_0-\delta\vartheta_0)-(\vartheta_4-\delta\vartheta_4)(\vartheta_0+\delta\vartheta_0)=2\{\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0\},$$

where the arguments are (u, v), and the δ denotes $u'\dfrac{d}{du}+v'\dfrac{d}{dv}$.

$\vartheta_5\vartheta_1+\vartheta_1\vartheta_5$, that is $\vartheta_5(u+u')\vartheta_1(u-u')+\vartheta_1(u+u')\vartheta_5(u-u')$, becomes simply $=2\vartheta_5\vartheta_1$, and similarly $\vartheta_{13}\vartheta_9+\vartheta_9\vartheta_{13}$ becomes $=2\vartheta_{13}\vartheta_9$; and the equation thus is

$$-2\alpha_1\gamma_1(\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0)+(\alpha_1\delta\delta_1+\gamma_1\delta\beta_1)\vartheta_5\vartheta_1+(-\alpha_1\delta\delta_1+\gamma_1\delta\beta_1)\vartheta_{13}\vartheta_9=0,$$

where the proper suffix 1 is restored to the α, $\delta\beta$, γ, and $\delta\delta$.

139. The equation shows that the differential combination $\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0$ is a linear function of $\vartheta_5\vartheta_1$ and $\vartheta_{13}\vartheta_9$, the coefficients of these products being of course linear functions of u' and v'; writing the equation

$$\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0=\mathrm{A}\vartheta_5\vartheta_1+\mathrm{B}\vartheta_{13}\vartheta_9,$$

we can if we please determine the coefficients in terms of the constants c', c'', c''', c^{iv}, c^{v}; viz., taking u, v indefinitely small, we have

$$\begin{aligned}
\vartheta_0&=c_0, & \delta\vartheta_4&=u'(c_4'''u+c_4^{\text{iv}}v)+v'(c_4^{\text{iv}}u+c_4^{\text{v}}v),\\
\vartheta_4&=c_4, & \delta\vartheta_0&=u'(c_0'''u+c_0^{\text{iv}}v)+v'(c_0^{\text{iv}}u+c_0^{\text{v}}v),\\
\vartheta_1&=c_1, & \vartheta_5&=\quad c_5'u+c_5''v,\\
\vartheta_9&=c_9, & \vartheta_{13}&=\quad c_{13}'u+c_{13}''v,
\end{aligned}$$

or substituting, and equating the coefficients of u and v respectively

$$\begin{aligned}
c_0(c_4'''u'+c_4^{\text{iv}}v')-c_4(c_0'''u'+c_0^{\text{iv}}v')&=\mathrm{A}c_1c_5'+\mathrm{B}c_9c_{13}',\\
c_0(c_4^{\text{iv}}u'+c_4^{\text{v}}v')-c_4(c_0^{\text{iv}}u'+c_0^{\text{v}}v')&=\mathrm{A}c_1c_5''+\mathrm{B}c_9c_{13}'',
\end{aligned}$$

which equations give the values of A, B.

140. Disregarding the values of the coefficients, and attending only to the form of the equation

$$\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0=\mathrm{A}\vartheta_5\vartheta_1+\mathrm{B}\vartheta_{13}\vartheta_9,$$

this is one of a system of 120 equations; viz.: referring to the foregoing table of the 120 pairs, it in fact appears that taking any pair such as $\vartheta_0\vartheta_4$ out of the upper compartment or the lower compartment of any column of the table, the corresponding differential combination $\vartheta_0\delta\vartheta_4-\vartheta_4\delta\vartheta_0$ is a linear function of any two of the four pairs in the other compartment of the same column.

Differential relation of u, v and x, y.

141. We have as before, in the two notations, the pairs

A . B		11 . 7	
C . DE		$\overline{5}$. 9	
D . CE		13 . 1	
E . CD		14 . 2	
F . AB		10 . 6	

and from the expressions given above for the four pairs below the line, it is clear that any linear function of these four pairs may be represented by

$$(a-b)\frac{1}{\theta}\{(\lambda+\mu y)\sqrt{\text{cdefa}_{,}\text{b}_{,}}+(\lambda+\mu x)\sqrt{\text{c}_{,}\text{d}_{,}\text{e}_{,}\text{f}_{,}\text{ab}}\},$$

where λ, μ are constant coefficients, and the factor $(a-b)$ has been introduced for convenience, as will appear.

We have consequently a relation

$$\sqrt{\text{aa}_{,}}\delta\sqrt{\text{bb}_{,}}-\sqrt{\text{bb}_{,}}\delta\sqrt{\text{aa}_{,}}=\frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\text{cdefa}_{,}\text{b}_{,}}+(\lambda+\mu x)\sqrt{\text{c}_{,}\text{d}_{,}\text{e}_{,}\text{f}_{,}\text{ab}}\},$$

where as before δ is used to denote $u'\frac{d}{du}+v'\frac{d}{dv}$, u' and v' being arbitrary multipliers, considering u, v as functions of x, y, we have

$$\frac{d}{du}=\frac{dx}{du}\frac{d}{dx}+\frac{dy}{du}\frac{d}{dy},$$

$$\frac{d}{dv}=\frac{dx}{dv}\frac{d}{dx}+\frac{dy}{dv}\frac{d}{dy},$$

and thence $\delta=\text{P}\frac{d}{dx}+\text{Q}\frac{d}{dy}$ if for shortness P, Q are written to denote $u'\frac{dx}{du}+v'\frac{dx}{dv}$ and $u'\frac{dy}{du}+v'\frac{dy}{dv}$ respectively.

142. The left hand side then is

$$=\mathrm{P}\left(\sqrt{\mathrm{aa}_{,}}\frac{d}{dx}\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\frac{d}{dx}\sqrt{\mathrm{aa}_{,}}\right)+\mathrm{Q}\left(\sqrt{\mathrm{aa}_{,}}\frac{d}{dy}\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\frac{d}{dy}\sqrt{\mathrm{aa}_{,}}\right);$$

the coefficients of P and Q are at once found to be

$$=-\tfrac{1}{2}\frac{(\mathrm{a}-\mathrm{b})\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}},\ -\tfrac{1}{2}\frac{(\mathrm{a}_{,}-\mathrm{b}_{,})\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}\ \text{respectively,}$$

or observing that $\mathrm{a}-\mathrm{b},=\mathrm{a}_{,}-\mathrm{b}_{,},=a-b$, the equation becomes

$$\mathrm{P}\frac{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}{\sqrt{\mathrm{ab}}}+\mathrm{Q}\frac{\sqrt{\mathrm{ab}}}{\sqrt{\mathrm{a}_{,}\mathrm{b}_{,}}}=-\frac{2}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}+(\lambda+\mu x)\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\};$$

or multiplying by $\sqrt{\mathrm{aba}_{,}\mathrm{b}_{,}}$ and writing for shortness $\mathrm{abcdef}=\mathrm{X}$, $\mathrm{a}_{,}\mathrm{b}_{,}\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}=\mathrm{Y}$, this becomes

$$\mathrm{a}_{,}\mathrm{b}_{,}\{\mathrm{P}+\frac{2}{\theta}(\lambda+\mu y)\sqrt{\mathrm{X}}\}+\mathrm{ab}\{+\mathrm{Q}\frac{2}{\theta}(\lambda+\mu x)\sqrt{\mathrm{Y}}\}=0.$$

143. There are, it is clear, the like equations

$$\mathrm{b}_{,}\mathrm{c}\{\mathrm{P}+\frac{2}{\theta}(\lambda'+\mu' y)\sqrt{\mathrm{X}}\}+\mathrm{bc}\{\mathrm{Q}+\frac{2}{\theta}(\lambda'+\mu' x)\sqrt{\mathrm{Y}}\}=0,$$

$$\mathrm{c}_{,}\mathrm{a}_{,}\{\mathrm{P}+\frac{2}{\theta}(\lambda''+\mu'' y)\sqrt{\mathrm{X}}\}+\mathrm{ca}\{\mathrm{Q}+\frac{2}{\theta}(\lambda''+\mu'' x)\sqrt{\mathrm{Y}}\}=0,$$

and it is to be shown that $\lambda=\lambda'=\lambda''$ and $\mu=\mu'=\mu''$. For this purpose recurring to the forms

$$\sqrt{\mathrm{aa}_{,}}\delta\sqrt{\mathrm{bb}_{,}}-\sqrt{\mathrm{bb}_{,}}\delta\sqrt{\mathrm{aa}_{,}}=\frac{a-b}{\theta}\{(\lambda+\mu y)\sqrt{\mathrm{cdefa}_{,}\mathrm{b}_{,}}\ +(\lambda\ +\mu x\ )\sqrt{\mathrm{c}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ab}}\},$$

$$\sqrt{\mathrm{bb}_{,}}\delta\sqrt{\mathrm{cc}_{,}}-\sqrt{\mathrm{cc}_{,}}\delta\sqrt{\mathrm{bb}_{,}}=\frac{b-c}{\theta}\{(\lambda'+\mu' y)\sqrt{\mathrm{adefb}_{,}\mathrm{c}_{,}}\ +(\lambda'\ +\mu' x\ )\sqrt{\mathrm{a}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{bc}}\},$$

$$\sqrt{\mathrm{cc}_{,}}\delta\sqrt{\mathrm{aa}_{,}}-\sqrt{\mathrm{aa}_{,}}\delta\sqrt{\mathrm{cc}_{,}}=\frac{c-a}{\theta}\{(\lambda''+\mu'' y)\sqrt{\mathrm{bdefc}_{,}\mathrm{a}_{,}}+(\lambda''+\mu'' x)\sqrt{\mathrm{b}_{,}\mathrm{d}_{,}\mathrm{e}_{,}\mathrm{f}_{,}\mathrm{ca}}\},$$

multiply the first equation by $\sqrt{\mathrm{cc}_{,}}$, the second by $\sqrt{\mathrm{aa}_{,}}$, and the third by $\sqrt{\mathrm{bb}_{,}}$, and add: the left hand side vanishes, and therefore the right hand side must also vanish identically.

144. But on the right hand side we have the term $\frac{1}{\theta}\sqrt{\mathrm{def}\,\mathrm{a}_,\mathrm{b}_,\mathrm{c}_,}$ multiplied into

$$(a-b)\mathrm{c}(\lambda+\mu y)+(b-c)\mathrm{a}(\lambda'+\mu' y)+(c-a)\mathrm{b}(\lambda''+\mu'' y),$$

and the term $-\frac{1}{\theta}\sqrt{\mathrm{d}_,\mathrm{e}_,\mathrm{f}_,\mathrm{abc}}$ multiplied into

$$(a-b)\mathrm{c}_,(\lambda+\mu x)+(b-c)\mathrm{a}_,(\lambda'+\mu' x)+(c-a)\mathrm{b}_,(\lambda''+\mu'' y),$$

and it is clear that the whole can only vanish if these two coefficients separately vanish. This will be the case if we have for λ, λ', λ'' the equations

$$\begin{aligned}(a-b)\lambda+(b-c)\lambda'+(c-a)\lambda''&=0,\\ c\ \ ,,\ \ +a\ \ ,,\ \ +b\ \ ,,\ \ &=0,\end{aligned}$$

and the like equations for μ, μ', μ''. The equations written down give

$$(a-b)\lambda : (b-c)\lambda' : (c-a)\lambda''=a-b : b-c : c-a$$

that is $\lambda=\lambda'=\lambda''$: and similarly $\mu=\mu'=\mu''$.

145. But this being so, the three equations in P, Q give

$$\mathrm{P}+\frac{2}{\theta}(\lambda+\mu y)\sqrt{\mathrm{X}}=0,\quad \mathrm{Q}+\frac{2}{\theta}(\lambda+\mu x)\sqrt{\mathrm{Y}}=0,$$

that is

$$\begin{aligned}u'\frac{dx}{du}+v'\frac{dx}{dv}&=-\frac{2}{x-y}(\lambda+\mu y)\sqrt{\mathrm{X}},\\ u'\frac{dy}{du}+v'\frac{dy}{dv}&=-\frac{2}{x-y}(\lambda+\mu x)\sqrt{\mathrm{Y}}.\end{aligned}$$

In these equations u' and v' are arbitrary; hence λ and μ must be linear functions of u' and v'; say their values are $=\varpi u'+\rho v'$, $\sigma u'+\tau v'$ respectively. We have therefore

$$\begin{aligned}\frac{dx}{du}&=-\frac{2}{\theta}(\varpi+\sigma y)\sqrt{\mathrm{X}}\ , & \frac{dx}{dv}&=-\frac{2}{\theta}(\rho+\tau y)\sqrt{\mathrm{X}},\\ \frac{dy}{du}&=-\frac{2}{\theta}(\varpi+\sigma y)\sqrt{\mathrm{Y}}\ , & \frac{dy}{dv}&=-\frac{2}{\theta}(\rho+\tau x)\sqrt{\mathrm{Y}},\end{aligned}$$

or, what is the same thing,

$$\begin{aligned}-\tfrac{1}{2}\theta\frac{dx}{\sqrt{\mathrm{X}}}&=(\varpi+\sigma y)du+(\rho+\tau y)dv,\\ -\tfrac{1}{2}\theta\frac{dy}{\sqrt{\mathrm{Y}}}&=(\varpi+\sigma x)du+(\rho+\tau x)dv,\end{aligned}$$

whence also

$$\sigma du+\tau dv=\ \ \tfrac{1}{2}\left(\frac{dx}{\sqrt{\mathrm{X}}}-\frac{dy}{\sqrt{\mathrm{Y}}}\right),$$

$$\omega du+\rho dv=-\tfrac{1}{2}\left(\frac{xdx}{\sqrt{\mathrm{X}}}-\frac{ydy}{\sqrt{\mathrm{Y}}}\right),$$

which are the required relations, depending on the square roots of the sextic functions X=abcdef, and $\mathrm{Y}=a_{,}b_{,}c_{,}d_{,}e_{,}f_{,}$ of x and y respectively; but containing the constants ϖ, ρ, σ, τ, the values of which are not as yet ascertained.

146. I commence the integration of these equations on the assumption that the values $u=0$, $v=0$ correspond to indefinitely large values of x and y. We have

$$\mathrm{X}=x^6\left(1-\frac{\mathrm{S}}{x}+\,.\,.\,\right),\ \mathrm{Y}=y^6\left(1-\frac{\mathrm{S}}{y}+\,.\,.\,\right),$$

where $\mathrm{S}=a+b+c+d+e+f$; and thence the equations are

$$\sigma du+\tau dv=\ \ \tfrac{1}{2}\frac{dx}{x^3}\left(1+\frac{\frac{1}{2}\mathrm{S}}{x}\,.\,.\,\right)-\tfrac{1}{2}\frac{dy}{y^3}\left(1+\frac{\frac{1}{2}\mathrm{S}}{y}\,.\,.\,\right),$$

$$\varpi du+\rho dv=-\tfrac{1}{2}\frac{dx}{x^2}\left(1+\frac{\frac{1}{2}\mathrm{S}}{x}\,.\,.\,\right)+\tfrac{1}{2}\frac{dy}{y^2}\left(1+\frac{\frac{1}{2}\mathrm{S}}{y}\,.\,.\,\right),$$

hence integrating

$$\sigma u+\tau v=-\tfrac{1}{2}\left(\frac{1}{x^2}-\frac{1}{y^2}\right)+\,.\,.\,.\,,$$

$$\varpi u+\rho v=\ \ \tfrac{1}{2}\left(\frac{1}{x}-\frac{1}{y}\right)+\tfrac{1}{8}\mathrm{S}\left(\frac{1}{x^2}-\frac{1}{y^2}\right),$$

and thence

$$\varpi u+\rho v+\tfrac{1}{4}\mathrm{S}(\sigma u+\tau v)=\tfrac{1}{2}\left(\frac{1}{x}-\frac{1}{y}\right)+\,.\,.\,.\,,$$

where the omitted terms depend on $\frac{1}{x^3}$, $\frac{1}{y^3}$ &c.

Hence neglecting these terms

$$\frac{\sigma u+\tau v}{\omega u+\rho v+\frac{1}{4}\mathrm{S}(\sigma u+\tau v)}=-\left(\frac{1}{x}+\frac{1}{y}\right),$$

an equation connecting the indefinitely small values of u, v, with the indefinitely large values of x, y.

147. From the equations $\mathrm{A}=k_{11}\omega\sqrt{a}$, $\mathrm{B}=k_{7}\omega\sqrt{b}$, taking (u, v) indefinitely small and therefore (x, y) indefinitely large, we deduce

$$\frac{c_{11}'u+c_{11}''v}{c_7'u+c_7''v}=\frac{k_{11}}{k_7}\,\frac{1-\frac{1}{2}a\left(\frac{1}{x}+\frac{1}{y}\right)}{1-\frac{1}{2}b\left(\frac{1}{x}+\frac{1}{y}\right)},$$

and hence substituting for $\frac{1}{x}+\frac{1}{y}$ the foregoing value, and introducing an indeterminate multiplier M, we obtain

$$c_{11}'u+c_{11}''v=\mathrm{M}k_{11}\{\varpi u+\rho v+\tfrac{1}{4}\mathrm{S}(\sigma u+\tau v)+\tfrac{1}{2}a(\sigma u+\tau v)\},$$

which breaks up into the two equations

$$c_{11}'=\mathrm{M}k_{11}\{\varpi+(\tfrac{1}{4}\mathrm{S}+\tfrac{1}{2}a)\sigma\},\quad c_{11}''=\mathrm{M}k_{11}\{\rho+(\tfrac{1}{4}\mathrm{S}+\tfrac{1}{2}a)\tau\}$$

and thence also

$$\begin{array}{llll}
c_7'=\mathrm{M}k_7\{\quad ,, & b\ \}, & c_7''=\mathrm{M}k_7\{\quad ,, & b\ \},\\
c_5'=\mathrm{M}k_5\{\quad ,, & c\ \}, & c_5''=\mathrm{M}k_5\{\quad ,, & c\ \},\\
c_{13}'=\mathrm{M}k_{13}\{\quad ,, & d\ \}, & c_{13}''=\mathrm{M}k_{13}\{\quad ,, & d\ \},\\
c_{14}'=\mathrm{M}k_{14}\{\quad ,, & e\ \}, & c_{14}''=\mathrm{M}k_{14}\{\quad ,, & e\ \},\\
c_{10}'=\mathrm{M}k_{10}\{\quad ,, & f\ \}, & c_{10}''=\mathrm{M}k_{10}\{\quad ,, & f\ \},
\end{array}$$

which twelve equations determine the coefficients $\varpi, \sigma, \rho, \tau$ in terms of the c', c'' of the odd functions 5, 7, 10, 11, 13, 14 ; and moreover give rise to relations connecting these c', c'' with each other and with the constants a, b, c, d, e, f.

148. It is observed that if as before

$$\eth=u'\frac{d}{du}+v'\frac{d}{dv},=\mathrm{P}\frac{d}{dx}+\mathrm{Q}\frac{d}{dy},$$

then, substituting for P, Q their values, we have

$$\eth=-\frac{2}{\theta}(\varpi u'+\rho v')\left(\sqrt{\mathrm{X}}\frac{d}{dx}+\sqrt{\mathrm{Y}}\frac{d}{dy}\right)-\frac{2}{\theta}(\sigma u'+\tau v')\left(y\sqrt{\mathrm{X}}\frac{d}{dx}+x\sqrt{\mathrm{Y}}\frac{d}{dy}\right),$$

$$=(\varpi u'+\rho v')\eth_1+(\sigma u'+\tau v')\eth_2,$$

if for shortness

$$\eth_1=-\frac{2}{\theta}\left(\sqrt{\mathrm{X}}\frac{d}{dx}+\sqrt{\mathrm{Y}}\frac{d}{dy}\right),\ \eth_2=-\frac{2}{\theta}\left(y\sqrt{\mathrm{X}}\frac{d}{dx}+x\sqrt{\mathrm{Y}}\frac{d}{dy}\right),$$

and then operating with $\eth$ on the equations $\mathrm{A}=\omega k_{11}\sqrt{ab}$, &c., we have for instance

$$\begin{aligned}\mathrm{A}\eth\mathrm{B}-\mathrm{B}\eth\mathrm{A}=\omega^2 k_{11}k_7\{&(\varpi u'+\rho v')(\sqrt{a}\eth_1\sqrt{b}-\sqrt{b}\eth_1\sqrt{a})\\&+(\sigma u'+\tau v')(\sqrt{a}\eth_2\sqrt{b}-\sqrt{b}\eth_2\sqrt{a})\},\end{aligned}$$

which is one of a system of 120 equations, the A, B being in fact any two of the 16 functions.

These are in fact nothing else than the foregoing system of 120 equations giving the values of the differential combinations $\vartheta_0\eth\vartheta_1-\vartheta_1\eth\vartheta_0$, &c., each as a sum of products of pairs of functions, only on the right-hand sides we have expressions such as $\sqrt{a}\eth_1\sqrt{b}-\sqrt{b}\eth_1\sqrt{a}$, &c., which present themselves as perfectly determinate functions of x, y: so that regarding $\varpi u'+\rho v'$, $\sigma u'+\tau v'$ as given linear functions of the arbitrary quantities u', v', there is no longer anything indeterminate in the form of the equations.

XXII. *Revision of the Atomic Weight of Aluminum.*

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Received March 13,—Read April 22, 1880.

Need for a redetermination of the atomic weight of aluminum.

There is probably no one of the so-called chemical elements equally abundant in nature with aluminum, and occurring in as numerous compounds, with regard to the atomic weight of which our knowledge has long rested upon so slender a foundation of accurate experiment. The following brief statement includes, I believe, all the determinations of this constant which are on record.

Former Determinations.

1. *Experiments of* BERZELIUS, 1812.—BERZELIUS* precipitated a solution of alum by addition of ammonia, dissolved the precipitate in sulphuric acid to saturation, filtered, concentrated the filtrate by evaporation, and threw down aluminum sulphate by alcohol. This salt was well washed with alcohol, to separate as far as possible any excess of acid, and was then heated in a platinum crucible over an alcohol lamp, weighing from time to time, until no further loss of weight occurred. The anhydrous sulphate so obtained was but slowly soluble in water on heating, but left no insoluble residue. 10 grms. of this salt was now raised to a higher temperature in a weighed platinum crucible, and strongly heated as long as any loss of weight could be detected. The residue of loose, light, white alumina found in the crucible weighed 2·9934 grms. Consequently the salt consisted of—

"Sulphuric acid" (SO_3)	70·066	or	100·000
Alumina	29·934	„	42·722
	100·000	„	142·722

Several small arithmetical errors have been made in the discussion of this single experiment of BERZELIUS, upon which for nearly half a century the value assigned to the atomic weight of aluminum may be said to have rested.

In the original paper it is calculated that if 42·722 parts of alumina contain 19·96

* GILBERT'S 'Annalen der Physik,' xl (1812), 260.

parts of oxygen (the atomic weight of sulphur being then taken $=201\cdot16$—$O=100$), 100 parts of alumina must include 46·726 parts of oxygen. This last number should be 46·7207. In a later paper* by the same author, it is calculated from the results of the above experiment, that 100 parts of "sulphuric acid" are saturated by 42·7227 parts of alumina, and that the earth contains 46·7047 per cent. of oxygen; the earth is assumed to be Al_2O_3, and consequently the atomic weight of Al is found $=171\cdot667$ ($O=100$), or 27·49 ($H=1$). These numbers, correctly calculated from the percentage of oxygen taken in this second paper, should read 171·167 and 27·39 respectively. Or, if the percentage of oxygen of the former paper, corrected as above, be taken as 46·7207, the atomic weight, for Al_2O_3, will be 171·057 ($O=100$) or 27·37 ($H=1$).†

Finally, if BERZELIUS' direct results of experiment be taken, and recalculated with STAS' atomic weights for O (15·96) and S (31·98), the atomic weight of aluminum, its oxide being assumed Al_2O_3, will be 27·237 in reference to that of hydrogen as unity.

BERZELIUS‡ also attempted to obtain a pure hydrate of aluminum by precipitating the sulphate and nitrate with ammonia, but found that highly basic salts only were thrown down. Using the chloride instead, a hydrate was obtained which, after being dried in the sun, gave only water on heating, but there was a little loss from mechanically-carried-over alumina. This sun-dried hydrate left 64·932 per cent. of alumina free from acid. BERZELIUS therefore calculates that 100 parts of anhydrous alumina had been combined with 54 parts of water—this amount of water containing 47·65 parts of oxygen; while the alumina contains, as shown by the above-quoted analysis of the sulphate, 46·726 parts of oxygen. He remarks: "I cannot affirm that either the determination of the amount of water or that of the oxygen in the alumina is sufficiently exact; both are, however, so far so as to sufficiently show us that alumina, like the preceding bases, combines with an amount of water whose oxygen is equal to that of the earth itself."

In the decomposition by heat of aluminum sulphate, as thus used to furnish data from which to calculate the atomic weight of the metal, the following possible sources of error may be noticed:—The hydrate precipitated by ammonia from a solution of (presumably) *potash* alum might carry down with it traces of fixed alkali, and this latter be retained in the sulphate afterwards prepared from the hydrate. The tendency of aluminum to form basic salts suggests the possibility of traces of sulphuric acid being lost in the preliminary drying over the simple alcohol lamp, even at a temperature at which possibly the last traces of water might not have been removed. I have found from my own experiments that a trace of basic sulphate may, on the other hand, be obstinately retained even after prolonged exposure to a very high tempera-

* POGGENDORFF'S 'Annalen der Physik u. Chimie,' viii. (1826), 187.

† A. C. OUDEMANS, Jr. (in his 'Historisch-kritisch Overzigt van de Bepaling der Æquivalent-Gewigten van twee en twintig Metalen;' Leiden, 1853), calculates, from BERZELIUS' figures, $Al=171\cdot02$,

‡ GILBERT'S 'Annalen,' *loc. cit.*

ture, when it might be assumed that pure alumina alone was left. As BERZELIUS himself says, ignited alumina rapidly absorbs moisture from the air, involving risk of error in determining its weight. Traces of the light pulverulent alumina are liable to be mechanically carried away during the decomposition of the sulphate. It is observable that all these sources of error, except the last, tend in the same direction, to make the atomic weight of aluminum come out too high.

2. *Experiments of* Sir HUMPHRY DAVY, 1812.—In Sir HUMPHRY DAVY's 'Elements of Chemical Philosophy'—published in 1812, the same year in which BERZELIUS' first paper on this subject appeared—it is stated* that no direct researches had then been made on the quantity of oxygen in alumina, but that, from some experiments by the author on the quantity of ammonia required to decompose saturated solutions of alumina in acids, "it would appear that the number representing alumina is about 48, and, supposing it to consist of one proportion of aluminum and one of oxygen, 33 will be the number representing aluminum." The details of the experiments in question are not given, and the combining proportions of all substances having been very imperfectly known at the time—the number 15 is taken above for oxygen—it is needless to say that this passage throws no light upon the exact atomic weight of aluminum.

3. *Experiments of* THOMSON, 1825.—THOMSON† attempted to deduce the number representing this atomic weight from, (*a*) analyses by himself and others of sundry natural aluminous silicates, (*b*) analyses of potassium alum, and (*c*) analyses of hydrates of aluminum. He concluded from all his experiments that the true number for alumina is 2·25 (O=1), and, taking alumina to be AlO, he made Al=1·25. This corresponds to Al=30, if O be assumed =16, and alumina Al_2O_3—a result which can only be viewed as a rough approximation to the truth, since THOMSON's methods were far from accurate, and his experimental results agree but poorly with each other.

4. *Experiments of* MATHER, 1835.—W. W. MATHER,‡ Assistant Professor of Chemistry at the United States' Military Academy, West Point, prepared anhydrous aluminum chloride by WOHLER's process, dissolved a weighed portion of it in water, added silver nitrate in excess, filtered off, dried and weighed the silver chloride formed, threw down excess of silver from the filtrate by hydrochloric acid, filtered again, evaporated this second filtrate and washings to dryness, ignited the residue, and weighed it as alumina. ·646 grm. of aluminum chloride gave 2·0549731§ grms. of silver chloride (yielding on reduction 1·548161 grm. of silver) and ·2975 grm. of alumina.

* Vol. iv., p. 263, of the 'Collected Works of Sir H. DAVY,' edited by his brother Dr. JOHN DAVY; London, 1840.

† THOMAS THOMSON, M.D., 'An Attempt to Establish the First Principles of Chemistry by Experiment,' vol. i., p. 285; London, 1825.

‡ SILLIMAN's 'American Journal of Science and Arts,' xxvii. (1835), 241.

§ The seven decimal places are given, notwithstanding the statement by the author himself that his balance could weigh easily $\frac{1}{500}$ grain, and was sensible to $\frac{1}{100}$ grain!

From the amount of silver chloride found and silver obtained from it in this one experiment, and from the atomic weights of silver and chlorine adopted by BERZELIUS and THOMSON respectively, MATHER calculated values for the atomic weight of aluminum ranging from 1·82274 to 1·85430 (O=1, and the formula of the chloride being taken as $AlCl_3$), or 29·16384 to 29·66880 for O=16; but from the amount of alumina obtained and the amount of aluminum therein (the latter deduced from the chloride taken for analysis minus the chlorine found), he calculated the atomic weight for aluminum as 1·3188017 (O=1) for alumina taken as Al_2O_3, or 21·1008272 for O=16. He does not seem to have been struck by the evidence of some error in his own work which these discrepant numbers afford, but suggested that the figures given by BERZELIUS for the aluminum and oxygen in alumina might have been accidentally inverted, which would explain the disagreement between himself and the great Swedish chemist. In reality it is pretty plain that MATHER's alumina was not pure, either from fixed matter of some kind left behind from the acids and wash water used, or from absorption of moisture before weighing. If his most direct result only be taken as the basis of calculation, namely, the weight of aluminum chloride used and silver chloride obtained from it, using STAS' numbers for chlorine (35·37) and silver (107·66), the atomic weight of aluminum found will be 28·778 for the formula $AlCl_3$.

5. *Experiments of* MALLET, 1857.—In 1857 the writer of this paper attempted to use metallic aluminum, which had not long before begun to be manufactured and sold, for the determination of the atomic weight. At the meeting of the British Association held in that year at Dublin,* he gave a brief account of his experiments, which had been made with the metal of commerce, containing, as he found, only from 93 to 96 per cent. of pure aluminum. The exact nature and amount of the foreign substances present, chiefly iron and silicon, having been determined, the crude metal was dissolved in hydrochloric acid, the solution precipitated by ammonia, and from the amount of alumina left from the precipitate on ignition, after allowing for the impurities, the atomic weight was deduced. The results obtained from a few experiments were not satisfactory enough to warrant any proposal to modify the then received number. The probability that this number needed correction was, however, pointed out, with reasons for such an opinion; the desirability of obtaining for the purpose of new experiments really pure metallic aluminum was noticed; and it was suggested that difficulties connected with the accurate determination of alumina by the method which had just been tried might make it eligible to determine instead the hydrogen given off during the solution of the metal in acid.

6. *Experiments of* DUMAS, 1858.—DUMAS† redetermined the atomic weight in question by dissolving in water known weights of aluminum chloride, and ascertaining the quantity of silver, used as nitrate, which was required in each case for precipitation of the chlorine. The aluminum chloride had been carefully prepared on the large

* 'Report of British Association Meeting at Dublin, 1857: Transactions of Sections,' p. 53.

† 'Annales de Chimie et de Physique' [3], lv. (1859), p. 151.

scale, then sublimed from iron turnings, and re-sublimed from aluminum filings. It sometimes still contained traces of iron. Each specimen to be used was sublimed for the last time from aluminum, in a stream of dry hydrogen, into a small glass tube, which was sealed at both ends before the lamp, and reserved for one of the analytical experiments. Of these there were seven. In each the weight of the sealed tube and its contents was taken, a drawn out end opened, and the weight quickly verified after equilibrium of pressure with the outside air had been thus established; the tube was introduced into water, and after solution of its contents the weight of the empty tube was determined. It does not appear from the published paper how the risk of mechanical loss from violent action of the chloride on the water was guarded against; from my own experiments with the bromide this appears to be a point requiring careful attention. Nor is it stated how the quantity of silver used was determined, whether by weighing the chloride of silver formed, by measuring the volume of a standard solution of silver nitrate, or by weighing off a little less metallic silver than would be required, converting this into nitrate, adding it to the aluminum chloride solution, and completing the precipitation with a measured quantity of dilute standard solution of silver nitrate. Nor is it mentioned how, if at all, the error due to slight solubility of silver chloride in the liquid from which it was precipitated is obviated—a point not ignored by DUMAS, as appears from another part of the same paper,* and afterwards very carefully examined by STAS.

The results of the seven experiments were as follows, the atomic weight or equivalent being calculated for the formula Al_2Cl_3, and on the supposition that Ag=108, and Cl=35·5 :—

					Atomic weight
I.—1·8786	of Al_2Cl_3	required	4·543	of Ag	=13·74
II.—3·021†	"	"	7·292	"	=13·85 (should be 13·86)
III.—2·399	"	"	5·802	"	=13·73
IV.—1·922	"	"	4·6525	"	=13·68
V.—1·697	"	"	4·1015	"	=13·77
VI.—4·3165	"	"	10·448	"	=13·68
VII.—6·728	"	"	16·265	"	=13·76
		Mean			13·744
		Should be			13·746 or 27·492 for $AlCl_3$

Recalculating this table for the formula $AlCl_3$, and using the atomic weights of STAS for silver (107·66) and chlorine (35·37), the figures of the last column become—

* Page 135.

† Contained traces of iron.

I.—Atomic weight of Al	. . . .	$=27{\cdot}447$
II. ,, ,, ,,	. . . . .	$=27{\cdot}696$
III. ,, ,, ,,	. . . . .	$=27{\cdot}435$
IV. ,, ,, ,,	. . . . .	$=27{\cdot}318$
V. ,, ,, ,,	. . . . .	$=27{\cdot}522$
VI. ,, ,, ,,	. . . . .	$=27{\cdot}327$
VII. ,, ,, ,,	. . . . .	$=27{\cdot}489$
Mean	. . . .	$27{\cdot}462$

It is to be remarked that the tendency of the chief causes of error connected with these experiments is in the same general direction. The presence of any iron in the chloride used; the action upon it, though but to a minute extent, of any trace of moisture in the hydrogen in which it was finally sublimed; any loss occurring with the fumes formed on introduction of the chloride into water; and the retention of traces of silver chloride in solution in the liquid from which the main mass of this compound had been thrown down; any or all of these would tend to diminish the quantity of silver chloride obtained, and therefore to make the atomic weight of aluminum appear greater than it really is. In discussing results which we owe to the labours of such experimenters as BERZELIUS and DUMAS, it is of course to sources of error likely to inhere in the method itself that attention should especially be given.

DUMAS also tried dissolving aluminum (containing iron and silicon in considerable quantity) in hydrochloric acid, adding nitric acid in excess, evaporating to dryness, igniting and weighing the alumina (and other oxides) left behind. From an analysis of the crude metal employed, so as to allow for the impurities present, and on the basis of three experiments made as above, he calculated the atomic weight as

$13{\cdot}74$ ($=27{\cdot}48$ for Al_2O_3—$O=16$)
$13{\cdot}87$ ($=27{\cdot}74$,, ,, — ,,)
$13{\cdot}89$ ($=27{\cdot}78$,, ,, — ,,)

but he was dissatisfied with these results, having found that the impurities of the metallic aluminum were unequally distributed throughout its mass, and having been unable to obtain the metal in a pure state. He considered the results furnished by the experiments with the chloride as accurate, and concluded that the atomic weight of Al is represented by the number $13{\cdot}75$ (for Al_2Cl_3)—this becomes $27{\cdot}5$ for $AlCl_3$ or Al_2Cl_6.

7. *Experiments of* TISSIER, 1858.—CH. TISSIER* prepared aluminum by reducing very pure fluoride of aluminum and sodium—probably cryolite, although this is not stated—by means of purified sodium in a carbon crucible, re-fusing the metal several times in order to free it from any of the flux which might have been retained. No

* 'Comptes Rendus des Séances de l'Acad. des Sciences,' xlvi (1858), p. 1105.

account is given of the means taken to prepare a carbon crucible free from iron and silicon; and to prevent destruction of the crucible by its material burning away during the fusion at high temperature of the aluminum salt with sodium. The metal was tested for iron by dissolving it in nitro-hydrochloric acid, evaporating to dryness with a large excess of nitric acid, and igniting the residue of alumina, which was observed to be of brilliant whiteness, while the addition of a solution containing a few thousandths of iron sufficed "to colour it very strongly red." Why the much more delicate tests available for iron in the original solution were not used does not appear.

As regards silicon, it is stated that "the solution of the metal by means of hydrochloric acid left no trace of silicon;" no mention is made of the solution having been evaporated to dryness, the residue remoistened with strong hydrochloric acid and dissolved in water in order to see whether silica was left. A portion of the solution obtained with nitro-hydrochloric acid was evaporated to dryness, the residue ignited, and the alumina so left was digested with a strong and boiling solution of ammonium nitrate. This solution was evaporated to dryness, and left a residue of sodium nitrate representing 0·135 per cent. of sodium in the metallic aluminum.

1·935 grm. of this aluminum was dissolved in hydrochloric acid, the solution evaporated with an excess of nitric acid until all chlorine was completely driven off, and the residue heated until the nitric acid was also completely removed and alumina only was left. This alumina weighed 3·645 grms. In the paper recording this single experiment the resulting atomic weight is not calculated, but the author simply points out that the number 14, which he says many chemists adopt as representing aluminum, must be too high, while 13 75, the number assigned by DUMAS, is in all probability accurate. In support of this view it is calculated that, if Al=14, the alumina obtained in the above described experiment should have weighed 3·590* grms.; whereas, with Al=13·75, its weight should have been 3·624 grms. In getting these figures O is taken =8.

But, if the minute quantity of sodium stated to have existed in the metal used be deducted, and allowed for as sodium oxide (aluminate) in the last weighed residue, and if the results obtained be calculated, for alumina $=Al_2O_3$, with STAS' number for oxygen (15·96), the atomic weight of aluminum will be represented by 27·068, a number much nearer to 27 than to 27·5 (13·75×2), the value assumed as most probably correct by TISSIER.

8. *Experiments of* TERREIL, 1879.—Lastly, about a year ago TERREIL† made a determination of the constant in question by passing hydrochloric acid gas over metallic aluminum, collecting and measuring the hydrogen evolved. He placed a known weight of aluminum in a tube of hard glass, the tube wrapped with foil so as to allow of its being made red hot. By one end a stream of well dried gaseous hydrochloric acid could be introduced, while a smaller tube extended from the other end and dipped into a vessel of water.

* This ought to read 3 594.

† 'Bulletin de la Société Chimique de Paris,' xxxi. (20 Fév, 1879), p. 153.

The air was first expelled from the apparatus by a current of dry carbon dioxide, and not until the gas passing through was capable of being completely absorbed by a solution of potash was the hydrochloric acid introduced, this latter itself freed from atmospheric air. The gas escaping from the tube was now collected in a graduated jar, and the temperature of the tube containing the aluminum was raised to a red heat. As soon as hydrogen ceased to come over, the gas in the jar was shaken up with potash to absorb any carbon dioxide which it might contain, and the volume was measured, and reduced by calculation to its equivalent under normal temperature and pressure. The aluminum chloride left in the tube was pulverulent and snow white.

No details are given of the method by which pure metallic aluminum was prepared, although this has been the great difficulty in the way of obtaining accurate results from experiments made with the metal as the starting point, nor is there any record of the tests applied to prove the purity of the metal used. The gas was collected over water, in which hydrogen is not altogether insoluble, and from which more or less of the gases of atmospheric air would be given off into the hydrogen. Nothing is said of the vapour of water, mixed with the hydrogen in proportion depending upon the temperature, having been removed, or its amount calculated and allowed for; though, as it is not likely that so obvious a precaution was neglected, it may be supposed that the potash spoken of as used to absorb any carbon dioxide left was either solid hydrate or so strong a solution as to have also removed most if not all the aqueous vapour.

The results of the single experiment reported were—

Weight of aluminum	·410 grm.
Volume of hydrogen collected at 11° and 768 m.m.	530 c.c.
Corresponding volume at normal temperature and pressure	508·2 c.c.
Weight of this hydrogen	·0455 grm.

from which the author calculates

$$\cdot 0455 : \cdot 4100 = 1 : 9 \cdot 01,$$

giving the atomic weight 13·515 (for Al_2Cl_6) or 27·03 (for $AlCl_3$).

In verifying the above calculation I have found as the result of reducing the volume of hydrogen from the given to normal temperature and pressure 514·85 c.c. instead of 508·2 c.c., but this, I am satisfied, arises from the number representing the pressure at the time of experiment being, doubtless by a printer's error, wrongly given as 768 instead of 758 m.m. With the latter figures the result is as recorded in the paper, and of course such atmospheric pressure is more frequently observed than that which appears in the above table.

It is pretty plain that from the researches which have been quoted we may reject those of DAVY, THOMSON, and MATHER as incapable of giving exact results, this being

either admitted by the authors themselves or shown by an examination of their methods and the inconsistency of their conclusions. Our knowledge of the atomic weight under consideration rests therefore upon the investigations of BERZELIUS, TISSIER, and TERREIL, each of whom made one experiment, and those of DUMAS, who made seven.

General results of former determinations.—If all the results be taken as I have re-calculated them, using STAS' atomic weights for the other elements concerned, and equal value be given to all, we shall have the following arithmetic mean -

BERZELIUS	27·237
DUMAS	27·447
,,	27·696
,,	27·435
,,	27·318
,,	27·522
,,	27·327
,,	27·489
TISSIER	27·068
TERREIL	27·030
Mean	27·357

If, however, the results of DUMAS, all depending on repetition of the same process, be viewed as possibly affected by some constant error, and be thrown together, taking into the calculation the mean only of his seven experiments, the general mean will be—

BERZELIUS	27·237
DUMAS (mean)	27·462
TISSIER	27·068
TERREIL	27·030
Mean	27·199

If we separate DUMAS' results, and take the mean of the other three, we get in contrast—

BERZELIUS	27·237
TISSIER	27·068
TERREIL	27·030
Mean	27·112
DUMAS (mean)	27·462

Or, if we throw together only the numbers obtained by TISSIER and TERREIL, which come nearest to each other, we have—

TISSIER		27·068
TERREIL		27·030
	Mean	27·049
	BERZELIUS	27·237
	DUMAS (mean)	27·462

Values now generally adopted for atomic weight of aluminum.

The number adopted in some of the more recent chemical handbooks, reports, &c., may be quoted as follows:—

GMELIN: 'Handbook of Chemistry' (Cav. Soc. Trans.)	27·4
WATTS: 'Dictionary of Chemistry,' First Supplement	27·4
W. A. MILLER: 'Elements of Chemistry,' 4th edit.	27 5
MEYMOTT TIDY: 'Handbook of Modern Chemistry'	27·5
FRANKLAND: 'Lecture Notes for Chemical Students'	27·5
THORPE: 'Quantitative Chemical Analysis'	27·26
'Agenda du Chimiste' (WURTZ' Laboratory, 1879)	27·5
'Annuaire du Bureau des Longitudes, 1876'	27·4
NAQUET: 'Principes de Chimie,' &c., 3e ed.	27·5
FITTICA: 'Jahresb. ub. d. Fortsch. d. Chemie, 1878'	27·4
ROSCOE u. SCHORLEMMER: 'Ausf. Lehrbuch d. Chemie' (Ger. ed.)	27·3
FRESENIUS: 'Anleit. z. quant. chem. Analyse,' 5te Aufl.	27·5
CLASSEN: 'Grundr. d. anal. Chemie' (quant.)	27·3
KOHLRAUSCH: 'Leitfaden d. prakt. Physik,' 2te Aufl.	27·4
MENDELEJEFF: Paper on the "Periodic Law" (transl.)	27·3
J. P. COOKE, Jr.: 'The New Chemistry'	27·5
J. D. DANA: 'System of Mineralogy,' 5th ed.	27·5
E. S. DANA: 'Text-Book of Mineralogy'	27·3

New experiments by the author.

During the last three years I have devoted a large part of my leisure time to a re-determination of this atomic weight, sparing no pains to attain as precise a result as possible, and aiming especially at the discovery, and as far as possible removal, of sources of error connected with the methods employed. The following general principles have been kept in view:—

1. That each process used should be as simple as possible, and should involve as little as possible of known liability to error.

2. That different and independent processes should be resorted to as the means of checking each other's results, even though it may fairly be assumed that one is more advantageous than another.

3. That each process should be carried out with quantities of material differing considerably from each other in successive experiments.

4. That only such other atomic weights should be involved as may be counted among those already known with the nearest approach to accuracy.

The most scrupulous care was taken in the purification and examination of all the reagents used, and as far as possible vessels of platinum or of hard porcelain were substituted for those of glass.

Means and method of weighing employed.—For the weighings an excellent balance, of BECKER's construction, was employed. It was in perfect order, carefully adjusted (especially as regards centre of gravity of beam with average load to be carried), and would bear safely 200 grms. in each pan, giving when thus loaded a deflection of the index to the extent of 1¼ division of the scale over which it moves for a difference of weight of ·0001 grm. All weighings were made by the well-known method of observing the vibrations of the index on either side the position of rest. In one series of experiments *absolute* weights were required, *i.e.*, real equality of weight between the quantities of matter dealt with and the standards of weight with which they were compared; in these cases the method of "double weighing" was made use of, so as to eliminate any error arising from inequality in length of the arms of the balance. In view of this need, in connexion with a part of the research, for absolute weights, directly comparable with those used by REGNAULT in his determination of the density of hydrogen, I applied to my friend J. E. HILGARD, Esq., in charge of the office of the United States' Coast Survey at Washington, for a comparison of a kilogramme with a weight of the same denomination belonging to the Coast Survey, the value of which latter weight is accurately known in terms of the original "kilogramme of the Archives" at Paris. He kindly had this comparison made, and sent me the results in detail, showing that my weight was 8·1 milligrammes heavier than the "star kilogramme" which is the standard of reference at Washington (both *in vacuo*), with an uncertainty of comparison not exceeding ·1 milligramme, while the "star kilogramme" is certified to as agreeing with the normal "kilogramme of the Archives" within 1·1 milligramme. I had already a 10-gramme weight, professedly normal, but, as it turned out, too light by a very minute fraction of its value, and with these two, checked against each other, a full series of comparisons was made of all the other weights to be employed, the specific gravity of each piece being determined before its final comparison as to weight, so that the real values might all be referred to a vacuum by calculation of the buoyancy in air. Determinations of the specific gravity of all materials and vessels which had to be weighed were also made, and, the barometer

and thermometer being observed at the time of each weighing, all weights hereafter mentioned in this paper represent real values *in vacuo.*

Three separate series of experiments were made, by methods to be presently stated. A fourth series was attempted, involving the conversion of metallic aluminum into oxide and determination of the amount of oxygen taken up, but this process was found to be attended with much difficulty from various causes, amongst others from the liability to loss by spirting if the metal were treated with acid in open or small vessels, from the necessity of transfer to such vessels for final ignition if larger ones were at first used, and from the appreciable solubility of the hydrate of aluminum if this were precipitated in order to avoid evaporation of the original solution. The few results obtained in this way agreed generally with those of the other methods, but varied among themselves within unsatisfactorily wide limits, and were manifestly not deserving of equal confidence. Hence the work was not pushed further in this direction.

First series of experiments.

Purification of ammonium alum.—Ammonium alum of commerce was dissolved in water, a very little nitric acid added, and the liquid boiled in large glass beakers by passing in a current of steam. When the solution had become cold a little ammonium ferrocyanide was added—a very small quantity sufficed to throw down the traces of iron present—and a little animal charcoal, previously well boiled with strong hydrochloric and nitric acid and thoroughly washed, was stirred in to aid in the subsidence of the minute amount of very finely divided Prussian blue which had been formed. The clear liquid was after about ten days drawn off, and evaporated until the larger part of the alum crystallised out on cooling. The crystals, of which there was obtained more than a kilogramme, were re-dissolved in hot water, and re-crystallised several times (throwing away all the mother liquors), the last time in a porcelain vessel, and with agitation, so as to obtain a granular crystalline powder, which was washed with cold distilled water.

Re-dissolving in water the so far purified alum, now much reduced in quantity, it became necessary to secure the exclusion of the metals of the fixed alkalies, lest their alums should exist in isomorphous admixture with the pure ammonium alum required. To this end aluminum hydrate was thrown down from the solution by addition of ammonia, using not quite enough of the reagent for complete precipitation. The precipitate was well washed with abundance of water, this tedious process being much expedited by the use of a siphon-filter delivering the liquid drawn off into a large flask connected with a powerful aspirator. The bulk of the precipitate was twice re-dissolved in pure hydrochloric acid (each time avoiding complete solution), thrown down again by ammonia, and again washed.

This hydrate was now dissolved in just the necessary amount of dilute and very carefully purified sulphuric acid, and just the proper amount added of ammonium sul-

phate prepared from the same sulphuric acid neutralised with ammonia. The quantities were determined by bringing the two solutions to known bulks, ascertaining by experiment on a sample of each how much of the respective salts was present, and measuring off the required volumes to be mixed. After concentrating the mixed solution by evaporation it was allowed to crystallise by cooling, and the crystallisation was repeated thrice, each time washing with a little cold water. On the last crystallisation pains were taken to regulate the rate of cooling so that as far as possible uniformly small crystalline grains were formed of about a millimetre in diameter, thus avoiding the liability of large crystals to contain cavities, in which mother liquor might be retained, and on the other hand securing the possibility of seeing with a lens, better than could have been done if the alum were in a still finer crystalline powder, that all the crystals as afterwards used were clear and transparent, and showed no signs of efflorescence.

It should be added that all the aqueous ammonia used as above for the precipitation of aluminum hydrate, and for its reconversion into alum, was recently and carefully prepared from ammonium chloride purified from alcoholic amines by boiling with nitric acid as recommended by STAS,* that the last crystallisations were effected from water which had been, in pursuance of the practice of the same chemist, distilled from potassium permanganate and hydrate, and that for these last crystallisations a large platinum dish was used, and care was taken that the solution was not allowed to boil, nor even to remain for any length of time near the boiling point, since I ascertained that ammonium alum, like simple ammonium sulphate, gradually gives off small quantities of ammonia on continued boiling of a strong solution. The very last crystallisation was carried out with only a sufficient quantity of the alum for a couple of the final experiments.

The salt thus purified was found to be free from any ascertainable content of foreign substances. It gave no trace of coloration in its solution when tested by a ferro-cyanide, tannic acid, &c., and by sulphuretted hydrogen and ammonium sulphide; and spectroscopic examination showed that the fixed alkaline metals and calcium were absent. Silver nitrate gave no indication of chlorine.

Ignition of ammonium alum.—Difficulties connected with this method.—I proposed to ignite a weighed quantity of this alum, whose distinct crystallisation gives it the advantage as to definiteness in the amount of water over the simple sulphate used by BERZELIUS, and to determine the weight of the aluminum oxide left behind, but careful examination of this process showed that two difficulties were to be feared.

In the first place, having rapidly dried the product of the last crystallisation by gentle pressure between folds of smooth filtering paper† free from loose fibre, portions were weighed off and exposed to the air at about 22° C., with the hope that before long a constant "air-dried" weight would be obtained. It had been previously ascer-

* Quoted in FRESENIUS' 'Zeitschrift für analyt. Chemie,' 6ter Jahrg., 4tes Heft., S. 423.

† This had been previously purified by ample washing with acid and water, and well dried.

tained that exposure over sulphuric acid led to large loss of water of crystallisation within a short time. It was found, however, that even in the air loss of weight went on for so long a time that it could not possibly be referred to mechanically adherent water only. It is true that this loss fell off very rapidly after the first hour or so, but it was impossible to decide precisely when it began to affect the water of crystallisation. In order to fully exhibit this I quote the following results obtained from a single large specimen kept very long on hand in a place carefully guarded against dust.

	Grms.
Original weight of alum after one hour's exposure to the air at 22°·5 C.	35·7456
Loss of weight in 1st twenty-four hours	·0088
,, ,, 2nd ,, ,,	·0025
,, ,, 3rd ,, ,,	·0017
,, ,, 4th ,, ,,	·0015
,, ,, 5th ,, ,,	·0010
,, ,, 6th ,, ,,	·0012
,, ,, 7th ,, ,,	·0009
,, ,, 8th ,, ,,	·0011
,, ,, 9th ,, ,,	·0007
,, ,, 10th ,, ,,	·0005

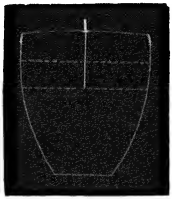
This series of weighings was carried on at longer intervals for six months more; the *monthly* loss was at first ·0087 grm., gradually fell off, became as small as ·0006 grm. in one month of cold weather when the temperature of the room was unusually low, and again rose, with warmer weather, to ·0012 grm. for the sixth and last month for which the weighings were continued. It was further found that, on placing some of the small crystals of the alum in a glass vessel deep enough to prevent mechanical loss, sensible loss of weight could be produced by the heat developed in simply crushing and pulverising the salt with a thick square-ended glass rod used as a pestle, and weighed along with the glass and its contents. While, therefore, it might be assumed as probable that mechanically adherent water would be got rid of within a time during which but a very minute quantity of water of crystallisation would be lost, a slight doubt is thrown over the exact formula of the salt as analysed, in reference to this component. Some of the ignition experiments were made with specimens which had been dried by longer exposure to the air than others, as will be noted hereafter.

In the second place it appeared that a minute trace of basic sulphate was retained by the alumina left after ignition at even a very high temperature. This could not be extracted by water, but was detectable by fusion with sodium carbonate free from sulphur, taking care to use an alcohol flame only. By moistening the alumina with a strong solution of pure ammonium carbonate, re-igniting, and repeating this treatment

a second time, it seemed to be possible to remove this source of error, as out of several specimens thus treated only one afforded a barely detectable trace of sulphate.

Details of method adopted.—The actual experiments on the ignition of the alum were carried out as follows. To render uniform the amount of atmospheric condensation on the surface of the vessel weighed, a very light glass bottle was specially made, with a delicately blown stopper, the latter carefully ground in and fitting quite air-tight; the bottle of a size to contain the platinum crucible in which the ignition was to be effected. The crucible was heated to bright redness, and while still quite hot was placed in a desiccator at some distance above a surface of recently distilled sulphuric acid. When cooled down to the temperature of the balance-room the crucible was as quickly as possible transferred to the weighing bottle, which was at once closed, and the combined weight of bottle and crucible was taken. The stopper was then removed for an instant, the cover of the crucible raised, and the quantity of alum desired, which had been roughly weighed off in a tube, having been poured in, occupying in no instance more than one-third depth of the vessel, cover

Fig. 1.



and stopper were replaced, and a second weighing gave, by the gain upon the first, the exact amount of alum used. Attached to the inner side of the crucible cover was a piece of rather stout platinum wire, which, when the cover was in place, ran down into the crucible in the line of its axis of figure, carrying two little diaphragms of platinum foil perforated with small holes (see fig. 1); such a diaphragm having been suggested and used by DUMAS* in his researches on atomic weights as the means of preventing any loss of solid particles which might otherwise be carried off mechanically from the substance ignited. To avoid inconvenience from the fusion of the alum in its water of crystallisation, and the swelling up of the salt to a bulky, porous mass, the heating was conducted very gradually. The platinum crucible was kept for an hour at 90° C., then for an hour at 100°, for an hour at 110°, another hour at 120°, a like time at 140°, and was then gradually brought to ignition over an argand alcohol lamp. It was then placed inside a larger platinum crucible, resting on a flat bit of unglazed porcelain at the bottom of the latter, and exposed to a gradually increased, and at

* 'Annales de Chimie et de Physique,' *loc. cit.*

last bright yellow heat in a gas furnace of FLETCHER's construction. This temperature was maintained for a full hour. On cooling down, the smaller crucible was taken out, the cover cautiously raised, and enough of a strong solution of ammonium carbonate introduced to moisten the alumina. Drying gently in a steam-bath, the crucible was re-ignited over an alcohol blast lamp, producing a strong red heat, and the addition of ammonium carbonate, drying, and ignition once repeated. As soon now as the crucible had ceased to be visibly red hot it was placed in the desiccator as at first, allowed to cool down to the temperature of the balance-room, quickly transferred to the weighing bottle, the stopper of which was inserted, and the final weighing was made while the alumina in the crucible was thus protected from absorption of moisture from the air. These experiments were carried out during a period of remarkably steady weather, with very little variation of atmospheric temperature, pressure, or moisture in the balance-room during the whole series.

Direct results of first series of experiments.—The results were as follows :—

A.—Alum dried by exposure to air for 2 hours at 21°–25° C.

I.—	8·2144	grms. of $(NH_4)_2Al_2(SO_4)_4.24H_2O$ left	·9258	grm. of Al_2O_3.
II.—	14·0378	„ „ „	1·5825	„
III.—	5·6201	„ „ „	·6337	„
IV.—	11·2227	„ „ „	1·2657	„
V.—	10·8435	„ „ „	1·2216	„

B.—Alum dried by exposure to air for 24 hours at 19°–26° C.

VI.—	12·1023	grms. of $(NH_4)_2Al_2(SO_4)_4.24H_2O$ left	1·3660	grm. of Al_2O_3.
VII.—	10·4544	„ „ „	1·1796	„
VIII.—	6·7962	„ „ „	·7670	„
IX.—	8·5601	„ „ „	·9654	„
X.—	4·8992	„ „ „	·5528	„

Second series of experiments.

Preparation and purification of aluminum bromide.—Aluminum bromide was prepared by the action of liquid bromine upon metallic aluminum of commerce, and was afterwards carefully purified. The first action is so violent that without special precaution the process involves some danger. In a first attempt a lump of aluminum weighing 15 to 20 grms. was dropped into a long-necked flask containing a considerable quantity of bromine. There was little action for a few moments, but as soon as it began vivid combustion took place, torrents of bromine vapour were driven forth, and after the flask had cooled the surplus metal was found to have been completely fused and had nearly melted its way through the glass.

By the following arrangement the bromide was prepared in large quantity and without any trouble. About 50 c.c. of bromine was placed in a large untubulated retort of hard Bohemian glass, the neck of the vessel standing vertically upwards, and an elongated piece of ingot aluminum, the upper end of which was firmly tied with aluminum wire to a glass rod, was cautiously dipped into the liquid and withdrawn as soon as violent action began. By alternately lowering and raising the glass rod the lower end of the metal was immersed in the bromine at intervals short enough to keep up the temperature of the latter and make the action practically continuous, while there was no actual ignition, and but little bromine vapour was lost. As soon as a considerable portion of this bromine had become converted into aluminum bromide the further action became manageable. The remainder of the main quantity of metal to be treated was now at once added in lumps of 10 to 20 grms. each; a long tube funnel with a glass stop-cock near the upper end was introduced into the neck of the retort, and liquid bromine was allowed to drip in at just such a rate as to keep up steady but not inconveniently violent action, taking care to keep the metal always covered. When the pieces of metal had nearly disappeared the supply of bromine was stopped, about 30 grms. more of aluminum was added in filings, the contents of the retort were digested for 4 hours at about 230° C., and the fluid portion was then decanted off from the insoluble residue into another (tubulated) retort. Most of the silicon was left undissolved as a brown amorphous powder. Most of the iron was converted into ferric bromide, which, during the continued heating, was in part broken up, leaving ferrous bromide instead. A little copper derived from one sample only of the aluminum used, was of course converted into bromide also.

More than a kilogramme of crude aluminum bromide being thus prepared, it was purified by repeated fractional distillations at carefully regulated temperature, using as a receiver in each case the retort to be next employed, and adding each time, except the last, a few grammes of aluminum filings. About a sixth of the whole amount was each time first distilled off and rejected as liable to contain silicon bromide, a little of this compound actually occurring in the earlier distillations; and another sixth was left behind, in order to retain the iron, which was separated with greater difficulty. After five distillations the bromide was obtained perfectly colourless, and boiling steadily at 263°·3 C. under 747 m.m. pressure. Specimens were dissolved in water, and carefully examined for iron, silicon, copper, and other conceivable impurities, but none could be found. As an additional precaution, the last distillation was effected in a slow stream of pure nitrogen, so as to avoid any formation of oxide or oxy-bromide of aluminum, the propriety of this being suggested by BERTHELOT's recent results* as to the thermic relations of aluminum to oxygen and the haloids, and the distillate was collected in three successive portions, the results of whose analysis will be separately given further on; they go to show that these three portions were sensibly identical. The individual specimens of pure bromide required were collected in little tubes of thin,

* 'Bulletin de la Société Chimique de Paris,' 20 Mars, 1879, p 263.

hard glass, previously closed in at one end, carefully dried, and sealed at the other as soon as the tube was nearly filled, the fused bromide having been introduced through a miniature tube funnel to avoid smearing the upper end of the collecting tube, and a new, perfectly dry funnel used each time. The weight of each collecting tube had been taken beforehand, and the piece of glass drawn off in sealing being washed, dried, and weighed, the weight of the sealed tube itself was known, the difference between this and the total weight of tube and contents giving the amount of bromide.

Experiments to determine the amount of bromide in this compound by precipitation with a silver solution have the advantage over those of DUMAS, above quoted, upon the chloride that, as STAS has shown,* silver bromide practically does not share with the chloride of this metal the slight solubility in the exactly neutralised liquid from which precipitation has been effected which renders difficult an exact determination of the amount of silver needed. Pursuing in general the course so carefully examined by STAS, the following were the details of the method employed.

Preparation of pure metallic silver.—Pure metallic silver was prepared by dissolving in nitric acid nearly pure silver already on hand, diluting largely, precipitating with pure hydrochloric acid, digesting the precipitate with aqua regia, washing thoroughly, and reducing the purified chloride in the liquid way with sodium hydrate (from metallic sodium) and invert-sugar (from perfectly pure and well crystallised cane-sugar boiled with dilute hydrochloric acid). The metal, after having been carefully tested, was fused by a jet of purified hydrogen mixed with rather less oxygen than necessary for perfect combustion. To avoid the necessity for any cutting up afterwards with steel tools, the pulverulent metal was divided into a number of little lots of various weight, and these were supported upon the surface of little blocks made by compressing pure sugar charcoal (from cane-sugar quite free from heavy metals) made into a paste with pure cane-sugar syrup and gradually drying and heating to redness.† The fused silver thus obtained was examined for occluded oxygen, following the method of DUMAS‡ in his recent experiments. It was supported upon a thin layer of pure lime in a hard glass tube, and heated to moderate redness in a SPRENGEL vacuum. The amount of oxygen given off was less than that obtained by DUMAS, doubtless owing to the fact that his silver was fused under nitre and borax, while mine was, as just stated, melted on a surface of carbon with no flux. He obtained at the rate of 57 c.c. of oxygen (for 0° C. and 760 m.m.) per kilogramme of silver, and in other experiments, prolonging notably the time of fusion, as much as 158 c.c. and 174 c.c. I obtained but 34·63 c.c. per kilogramme, and in another experiment made by Mr. SANTOS, then a student in this Laboratory, the silver having been fused upon ordinary wood charcoal, but 30·12 c.c. was given off. All the silver used in the atomic weight determinations

* 'Comptes Rendus,' 73, 998 'Annales de Chimie et de Physique' [5], 3, 289.

† This form of support had the advantage that if any particles of carbon should be mechanically enclosed in the silver they would be readily seen on solution of the latter in nitric acid.

‡ 'Comptes Rendus,' 86, 65. 'Chem. Centralblatt,' 27 Febr., 1878, S. 138.

was, in separate portions, heated in the SPRENGEL vacuum as long as any gas was expelled, and, having been treated with pure hydrochloric acid to remove any possibly adhering particles of lime, the granules were finally washed with pure water, dried, and kept for use in a glass stoppered bottle. An approximate calculation having been made of the quantity of silver which would be required to precipitate each of the specimens of aluminum bromide, an amount less than this by something under a decigramme was dissolved in nitric acid in a strong flask closed with a stopper, which was carefully opened when cold, and the contents were somewhat further diluted with water. The amount of specially purified nitric acid used was apportioned so as to leave the smallest possible excess after solution of the metal.

Precipitation of silver bromide—Details of method used.—To avoid the danger of losing aluminum bromide when it was brought in contact with water, the action being quite violent and attended with dispersion of white fumes, each one of the sealed thin glass tubes containing the bromide was, when the time came for using it, cautiously marked with transverse scratches at intervals of about half-an-inch by means of a writing diamond, and a strong glass bottle with a very well ground stopper having had a sufficient quantity of pure water placed at the bottom, the tube was broken across at the uppermost scratch, above the surface of the bromide, the empty point was dropped into the bottle, and the rest of the tube carefully lowered by means of a loose fitting spiral of platinum wire held sideways, so as to rest by the closed end on the bottom of the bottle without allowing the water to reach the bromide until the stopper had been inserted and tied down. By now gently inclining the bottle the water was brought very gradually into contact with the aluminum bromide, without dangerously violent action and without possibility of loss.

As soon as solution was complete and the bottle had cooled down the stopper was removed, any liquid adhering to it washed back into the bottle, and a stout glass rod with square end was used to gently crush the tube to small fragments, as otherwise its contents could not have been brought fairly into contact with the silver solution, since the interior of the tube would have become plugged up with silver bromide. The scratches previously made upon the glass rendered it easy to break it up without any splashing, and the rod was then well washed with pure water allowed to run directly into the bottle. The silver nitrate solution destined for this particular specimen was now washed out of the flask in which it had been shortly before prepared into the bottle containing the bromide, the stopper was again inserted, and the bottle was vigorously shaken as in the usual GAY-LUSSAC silver assay. The precipitation of the bromine was completed with a very carefully adjusted solution of silver nitrate, containing 1 milligramme of silver per cubic centimetre, and delivered from a burette* reading clearly to $\frac{1}{20}$th c.c. The correspondence in capacity of the burette and measuring flask used was well ascertained. It was at first intended to

* This burette was simply drawn down to proper bore at the bottom, and the flow of the liquid was regulated by the admission of air through a well ground stop-cock at the top.

use, for some at least of the experiments, *an excess* of silver, filter off the silver bromide formed, and determine silver in the filtrate by VOLHARD's method* with a standard solution of sulpho-cyanate; but this plan was abandoned as less simple, and probably requiring further investigation in regard to inherent sources of error and limits of accuracy. The completion of the reaction was therefore ascertained simply by very cautious addition of the standard silver solution until all trace of turbidity ceased, verifying the result by counter test with a drop of very dilute solution of potassium bromide. By letting the bottle stand for a little while after each addition of silver solution and shaking, and then tilting it to one side so as to bring the upper portion of the liquid above the line it had before occupied on the glass, letting the new drop fall gently in, an exceedingly slight cloud was easily seen.

It should be added that the silver globules and the tubes of aluminum bromide, after final cleansing and drying, were handled only with forceps, so as to avoid any risk of traces of chlorides being taken up from the fingers, and due attention was given to the freedom of the laboratory atmosphere from hydrochloric acid or chlorine in other volatile forms.

Direct results of second series of experiments.—The results of the experiments made in this way were—

A.—Aluminum bromide from first portion of last distillate.

I.—	6·0024 grms. of $AlBr_3$ required	7·2793 grms. of Ag for precipitation		
II.—	8·6492 ,, ,,	10·4897 ,, ,,		
III.—	3·1808 ,, ,,	3·8573 ,, ,,		

B.—Aluminum bromide from second portion of last distillate.

IV.—	6·9617 grms. of $AlBr_3$ required	8·4429 grms. of Ag for precipitation.
V.—	11·2041 ,, ,,	13·5897 ,, ,,
VI.—	3·7621 ,, ,,	4·5624 ,, ,,
VII.—	5·2842 ,, ,,	6·4085 ,, ,,
VIII.—	9·7338 ,, ,,	11·8047 ,, ,,

C.—Aluminum bromide from third portion of last distillate.

IX.—	9·3515 grms. of $AlBr_3$ required	11·3424 grms. of Ag for precipitation.
X.—	4·4426 ,, ,,	5·3877 ,, ,,
XI.—	5·2750 ,, ,,	6·3975 ,, ,,

Third series of experiments.

Preparation of pure metallic aluminum.—A further supply of pure aluminum bromide having been made as above described, it was used to prepare metallic aluminum by reduction with sodium.

* FRESENIUS' 'Zeitschrift für analyt. Chemie,' 18ter Jahrg., 2tes u. 3tes Heft., S. 271.

With a view to render the bromide more manageable while in contact with the air, it was decided to unite it with an alkaline chloride, as the aluminum and sodium chloride is used in the ordinary process of DEVILLE, and in order to keep down the melting point, a mixture of both potassium and sodium chlorides was employed. The two alkaline salts were separately purified, carefully tested, well dried in platinum vessels, mixed in the proportion of one molecule of each, the mixture fused in lots of about 250 grms. in a platinum dish over a gas furnace, poured out into another such dish standing on a block of cold iron, the thin cake gently crushed to a coarse powder while still warm in a mortar of hard glazed porcelain (from which, it was afterwards proved by examination, no silica was taken up), and the pulverised product kept until needed in a well stoppered bottle. Separate portions of the purified aluminum bromide having been in the last distillation collected in tared glass flasks and weighed, a quantity of the above mixture of alkaline chlorides was weighed off for each corresponding to 1 molecule $(K+Na)Cl$ for 1 molecule $AlBr_3$, and the flask having been heated until the bromide was fused, the potassio-sodic chloride was cautiously added. Very marked rise of temperature occurred, so that in a first attempt, using considerable quantities of the materials and mixing them abruptly, since such an effect had not been foreseen, the flask was violently cracked, and torrents of aluminum bromide vapour were driven off. This evolution of heat is interesting as evidence of chemical combination taking place, not only between two chlorides or two bromides, but between a chloride and a bromide. The fused mass was on cooling crushed to small fragments and coarse powder and preserved in a well-stoppered bottle. The precaution was not omitted of testing for any evidence of impurity derived from the flasks or mortar, but with negative result. The material thus prepared did not fume in the air, was sufficiently slow in taking up atmospheric moisture to be managed without difficulty, and fused on re-heating to about 130° C. The sodium to be used in decomposing it was in large ingots, which when needed were wiped to remove naphtha, the outer crust cut off with a knife, the large pieces roughly weighed, and the proper quantity rapidly cut up into small fragments without again moistening with naphtha.

The great difficulty in the way of obtaining pure metallic aluminum consists in obtaining crucibles of suitable material, especially such as shall not yield either iron or silicon. M. TISSIER* seems to have been more fortunate than I in the use of carbon vessels. The crucibles of hard carbon which I had on hand, purchased in Germany, contained both the above-named impurities, which were taken up in no small quantity by the aluminum; and I failed in sundry attempts to make crucibles of purer carbon, or to use this substance as a lining, the carbon either burning away, crumbling up, or permitting the fused materials to pass through its pores or through cracks in the mass. I at last succeeded in adapting to my purpose alumina itself, sufficiently cemented together by sodium aluminate. I was indebted to HENRY

* 'Comptes Rendus,' *loc. cit.*

PEMBERTON, Esq., Vice-President of the Pennsylvania Salt Manufacturing Company at Natrona, Pennsylvania, and to W. N. RICHARD, Esq., of the same works, for an abundant supply of aluminum hydrate, such as is thrown down by a stream of carbon dioxide from solution of sodium aluminate in the process of making soda from cryolite. This was not absolutely free from iron, but one lot contained traces only of this metal, insufficient, as it turned out, to contaminate the aluminum to be made in contact with it. The hydrate was strongly heated in well covered crucibles until it ceased to give off water; the alumina which was left required then to be guarded from the air, as it readily took up moisture again. All attempts to use it mixed with water and any unobjectionable cementing material to a plastic mass failed from excessive shrinkage and crumbling, but by mixing it in the dry state with dry sodium aluminate better results were obtained. The sodium aluminate had to be specially prepared, as that made (for soap boilers' use) at Natrona from cryolite contained too much iron. The dry mixture was pressed into wooden moulds, and three or four crucibles thus made having been very slowly and cautiously heated up in a gas furnace, stood fairly the necessary temperature of the reduction of aluminum, although they were very fragile. On the whole, however, it was found best to use a highly aluminous BEAUFAYE crucible, with a thick lining of this mixture of dry alumina and sodium aluminate well rammed in and very gradually heated. There were some failures from cracking of the crucibles or linings, and whenever the slightest contact of the metallic aluminum with the outer crucible occurred, silicon and generally iron were sure to be found in the metal. With successful prevention of this by an adequately thick and perfectly continuous lining there was much difficulty in securing a sufficiently high temperature in the interior, since the conducting power for heat of the alumina linings seemed to be quite low. This led to much loss of sodium by combustion, and but a very small yield indeed of really pure aluminum was secured. When the crucible had been heated up ready for the reduction, a small quantity of the pulverised mixture of potassium and sodium chlorides was thrown in; soon afterwards the aluminum bromide (which had been fused with the alkaline chlorides) with the proper quantity of sodium was introduced,* and as soon as the violent reaction was over a further portion of the mixed alkaline chlorides.

Only the large, well-fused globules of aluminum were picked out; these were re-fused once or twice before a blowpipe flame upon a support of alumina, to free them from any possible remains of the flux; any trace of oxide was detached by acting slightly upon the surface with pure hydrochloric acid, and the globules were then well washed with water, and dried by a gentle heat. Specimens cut from different portions of the globules were carefully tested, particularly for silicon, iron, sodium, and

* Aside from the question of expense there is some advantage in using aluminum bromide for making the metal, on account of the low melting-point of the sodium bromide which is left. CARNELLEY'S recent determinations (Chem. Soc. Journ., July, 1878, pp. 279, 280) make the melting-point of sodium bromide 708° C., while that of the chloride is 772°, and that of the fluoride above 902°.

potassium; and a sufficient quantity of the metal for the intended experiments was found to yield no appreciable trace of impurity. The surface of the specimens to be used which had touched the cutting pliers was again cleansed by acid and water. This pure aluminum did not differ much in physical character from the ordinary metal of commerce; it seemed, however, to be somewhat whiter, was distinctly softer, and had a little higher density, the mean of three closely-agreeing determinations made at 4° C. giving the number 2·583 as referred to water at the same temperature.

Production of hydrogen by aluminum, and measurement of the gas.—Details of method used.—The pure metal was used for the determination of its atomic weight by acting upon a known weight of it with a strong solution of sodium hydrate and determining the amount of hydrogen evolved. The advantage of using an alkali rather than hydrochloric acid, as in TERREIL'S experiment above quoted, lies in the non-volatility of the former, only vapour of water having to be separated from the hydrogen, while sulphuric acid is not available on account of the resistance to its action of aluminum. In 1863 Fr. SCHULZE* proposed to measure the volume of hydrogen given off by the action of an alkaline solution on commercial aluminum as the means of approximately deciding on the comparative purity of different specimens. The nature of the reaction was established by preliminary experiments, which proved to me that normal sodium aluminate alone is formed, so that each atom of aluminum liberates three atoms of hydrogen. The sodium hydrate was prepared from metallic sodium, and was used in the form of a solution so strong as scarcely to lose a sensible amount of water by the passage through it of a dry gas at common temperature—such an alkaline solution, so far as strength is concerned, as would be used to absorb carbon dioxide in an organic analysis. The quantity taken for each experiment was but a few cubic centimetres, and but little beyond the exact amount required for the solution of the metal; a small excess was, however, always allowed, so that the action might not become very languid towards the end. This strong alkaline solution was prepared with water which had been boiled to expel air, and the solubility in it of hydrogen was ascertained to be so small that any correction on this account would have fallen within the limits of inevitable error, and might be safely neglected.

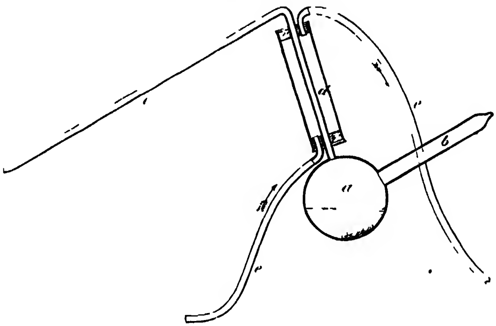
To secure accurate measurement of a somewhat large volume of hydrogen, two stout flasks were selected, one holding about a litre and the other about half as much, and with narrow necks of rather more than usual length; and on the necks a simple millimetre scale was marked. One of these flasks having been carefully filled with mercury and inverted over the mercurial trough, the hydrogen was collected in it, such a quantity of aluminum being used in each experiment as previous trials had shown would yield gas enough to bring the level of the mercury within the range of the scale on the neck. To thus obtain such a volume of gas as could be accurately measured in the narrow part of the flask it was necessary to note in advance roughly the prevailing

* FR. SCHULZE, "Die gasvolumetrische Analyse," S. 18, quoted in v. WAGNER, 'Jahresbericht,' u. s. w., 1864, S. 23.

temperature and pressure, and to take advantage of a time when these were not undergoing much change. The level of the mercury having been noted on the millimetre scale, the corresponding volume was afterwards determined by calibration with mercury, weighed in in portions of about half a kilogramme.

The little piece of apparatus used for the solution of the aluminum is shown in fig. 2, in which (*a*) represents a glass bulb of about 65 m.m. diameter; (*b*) a tube connected therewith, originally open at both ends, and about 12 m.m. diameter and 175 m.m. long, but afterwards reduced in length to about 100 m.m. by drawing off and sealing, as shown in the figure; (*c*) is the much smaller tube for carrying off the gas produced; (*d*) is a water jacket like that of the common LIEBIG'S condenser, formed by a piece of larger tube surrounding the ascending limb of (*c*), and put in place before (*c*) was bent; while (*e, e*) represent small indiarubber tubes for circulating a current of water through (*d*).

Fig 2



Each experiment was made as follows: The proper quantity of strong solution of sodium hydrate, its volume accurately measured, having been introduced into the bulb by means of a little tube funnel passed through (*b*), taking care to leave the surface of the latter clean, the aluminum (usually in a single piece of elongated shape) was passed into (*b*), held nearly horizontal, so that the metal did not slip down into the bulb, but rested 40 or 50 m.m. from it. (*b*) was now drawn off and sealed with a well-rounded end. The bulb was touched for a moment or two with the hand, so as to expel a very little air, and the outer end of the small tube (*c*) was introduced into the mercury of the trough, taking care that (*b*) was still kept in such a position as to prevent the aluminum coming in contact with the alkaline solution. After a sufficient lapse of time for the apparatus to have acquired the temperature of the room, the barometer and thermometer and the difference of level of the mercury in the trough and in (*c*) were read off;* so that, knowing the volume of alkaline solution introduced

* All readings were of course made from a distance with the aid of a small telescope.

and of aluminum (the latter from its weight), calibration of the bulb and tubes after the experiment was over completed the data necessary to determine the volume of air which the apparatus contained at the beginning. The aluminum was now made to slide down into the bulb, the end of the gas-delivery tube (*c*) having been brought under the mouth of the measuring flask. Over-rapid evolution of hydrogen and any considerable rise of temperature were prevented, partly by tilting the bulb so that the little piece of aluminum rested against one side and exposed but a part of its surface to the action of the liquid, and partly by cooling the outside of the bulb with water; while, on the other hand, it sometimes became necessary to gently warm the liquid towards the end of the experiment. To guard against more than traces of aqueous vapour being carried away with the hydrogen, a current of ice-water was kept up through (*d*). As soon as the last of the aluminum had disappeared, leaving the liquid quite clear, (c) was brought up into a nearly vertical position, and the apparatus left to itself until the temperature of the room had been attained. The barometer and thermometer, height of mercury in (*c*) above that in trough, and level of mercury in neck of measuring flask (after the last traces of moisture had been removed from the hydrogen by means of a stick of caustic potash), with its height above that in trough, were now read and recorded. Lifting (*c*) straight up from the trough, the mercury in this tube was got out by running a wire up and down in it, and inverting it, the whole of the remaining space in (*a*), (*b*), and (*c*) was filled up with alkaline lye of the same strength with that already contained, this liquid being run in from a graduated burette through a slender tube funnel, and the volume used noted, so as to show how much liquid had been already present. The apparatus being now emptied, washed out, and calibrated (with water, instead of mercury, on account of the difficulty of getting the interior quite dry), the volume of gas remaining in it at the close of the experiment was had from the difference between the total capacity (to the level of the mercury in (*c*)) and the volume of liquid which the bulb had contained at the close of the experiment, these taken together with the data for pressure and temperature.

On account of slight rise of temperature during the solution of the metal, the volume of hydrogen left in the bulb and tubes was always less than the air in the same at the beginning; and, after reduction to normal temperature and pressure, the difference had to be subtracted from the volume of gas collected in the flask.

In order to connect the weight of the aluminum with the weight of the hydrogen, the latter being obtained from its observed volume and REGNAULT's determination of its density, it was necessary that the weight of the metal should be absolute, or in terms of equal value with those used in REGNAULT's researches; hence, as has been already stated, the weights used were such as had had their real value determined, and the precaution of double weighing was applied. The quantities of metal used being small, the centre of gravity of the balance beam was so adjusted as to give great sensitiveness.

In calculating the weight of the hydrogen from its volume, the difference in the

value of the force of gravity at Paris and at the University of Virginia had to be taken into account. In view of the difference of latitude and elevation above the sea this constant is, in C.G.S. units,

For Paris	980·94
For University of Virginia . . .	979·95

and, applying the difference in nominally normal pressure at the two places, REGNAULT'S value for the weight of a litre of hydrogen at 0° C. and 760 m.m., ·089578 grm., becomes ·089488 grm.

In the experiments made in this way the only assignable cause of constant error, tending to affect in a particular direction the atomic weight deduced from them, seems to be the retention in solution of traces of hydrogen by the alkaline liquid in the bulb. The tendency of this is, of course, to make the atomic weight of aluminum appear greater than it should be, but I am satisfied that the possible extent of such error must be excessively minute, inappreciable within the limits of error of observation.

Direct results of first set of third series of experiments.—The results obtained were as follows:—

A.—Hydrogen by volume at 0° C. and 760 m.m.

I.— ·3697	grm. of Al	gave	458·8	c.c.	=·04106	grm. of H.
II.—·3769	„	„	467·9	„	=·04187	„
III.—·3620	„	„	449·1	„	=·04019	„
IV.—·7579	„	„	941·5	„	=·08425	„
V.—·7314	„	„	907·9	„	=·08125	„
VI.—·7541	„	„	936·4	„	=·08380	„

Second set of experiments of third series.—Hydrogen collected and weighed as water. —*Details of method used.*—As the collection and accurate measurement of larger quantities of hydrogen would be difficult from the great weight of mercury to be dealt with, while it seemed desirable to repeat these experiments with a larger amount of aluminum, an arrangement was adopted for burning the hydrogen and weighing it as water.

A bulb like that described above was used for the reaction between the aluminum and solution of sodium hydrate, another small tube (which may be designated as (x)) being connected, however, with the bulb, for the purpose of sweeping out air at the beginning of the combustion and the remainder of the hydrogen at the end by a current of other gas. The gas from the delivery tube (c) was carried through a series of four drying tubes, containing in the former two pumice stone soaked with pure sulphuric acid of full strength, and in the other two well dried asbestos and loose, woolly phosphorus pentoxide; thence it passed through a long combustion tube, filled for the first two-fifths with pure, finely granular cupric oxide, which had been recently ignited, and for two-fifths more with turnings of electrotype copper oxidised on the

surface, while the last one-fifth of its length remained at first unoccupied, to be filled later as will be described; the vapour of water being finally collected beyond this combustion tube in a single light drying tube of calcium chloride, one of sulphuric acid on pumice, and one of phosphorus pentoxide.

The whole of the apparatus having been put together, with the exception of the three final drying tubes, for which at first a single unweighed calcium chloride tube was substituted, with sodium hydrate solution in the bulb, and metallic aluminum in the tube (b), a slow stream of carefully purified and well dried air was passed through from (x) for some time, while the combustion tube was heated to low redness. Having allowed it to cool down to the temperature of the room, the stream of air from (x) was replaced by one of dry nitrogen; a plug of soft copper turnings with bright unoxidised surface was introduced into the further end of the combustion tube, so as to fill up the unoccupied fifth of its length, and it was again brought to and kept at a red heat. After the nitrogen had passed through at this temperature for about twenty minutes, the three drying tubes for the collection of the water to be formed, having been standing for some time in the balance case, were accurately weighed, and connected with the further end of the combustion tube, the previously used and unweighed calcium chloride tube being removed. The tube (x) having been closed, the tube (b) was now tilted so as to make the aluminum slip down into the alkaline liquid, and as in the experiments already described, the rate at which the hydrogen was evolved was controlled by inclining the bulb so as to vary the extent of surface of the metal attacked, and by cooling the outside of the bulb with water.

As soon as the metal was all dissolved, nitrogen was again introduced by (x) to sweep out the remaining hydrogen, limiting the quantity of the former gas to such an amount as was thought necessary for this purpose. This nitrogen was then in turn replaced by pure and dry air, which was passed through the apparatus until the surface of the copper which had been reduced was reoxidized, this being done to avoid any risk of occluded hydrogen being retained, while the nitrogen had served to obviate the danger of explosion. The drying tubes were then finally removed from the further end of the combustion tube, and weighed after exposure to the atmosphere of the balance case long enough to permit the surface of the glass attaining a constant condition. In both weighings the reduction to equivalent weights *in vacuo* was duly attended to, the weights and densities of all the materials making up the drying tubes and their contents having been previously ascertained. The last tube of the set was weighed separately, as was the last of the drying tubes connected with the reaction bulb, and it was found that, there being no increase of weight on the part of either of them, the absorption of aqueous vapour was sensibly complete.

Although all the precautions I could think of were taken in these experiments, the well known difficulty of absolutely excluding moisture, of which every joint to the apparatus becomes a possible source, so that in ordinary organic analysis the amount of hydrogen found may be expected to come out rather above than below the truth,

leads to the suspicion that such constant error as may have been involved tended in this direction, and if so that the resulting atomic weight would be made to appear somewhat too low. The extent of error from this cause, however, if it existed at all, must have been extremely small.

Direct results of second set of experiments of third series.—The results of these last experiments were—

B.—Hydrogen weighed as water.

I.—2·1704	grms.	of Al gave	2·1661	grms.	of H_2O.
II.—2·9355	"	"	2·9292	"	"
III.—5·2632	"	"	5·2562	"	"

Calculation of results.

In calculating the atomic weight of aluminum from the data furnished by the above described experiments, the atomic weights assumed for the other elements involved are those which result from the researches of STAS and the previous investigation by DUMAS and STAS of the composition of water, namely—

$$O=15·961 \qquad S=31·996 \qquad N=14·010$$

In regard to silver and bromine a difficulty arises from the fact that the relation between these elements was determined by STAS with metallic silver which, as DUMAS* has pointed out, contained in all probability occluded oxygen. It appears from DUMAS' experiments and from mine, that the quantity of oxygen which may be so retained varies with the conditions under which the metal is fused, and it is impossible now to ascertain precisely how much was present in that used by STAS, while the correction to be applied on this account, though small, is not inappreciable in its effect upon the atomic weight of the aluminum. Omitting to extract in the SPRENGEL vacuum the occluded oxygen from the silver used in my experiments would not have secured identical condition of the silver with that of the metal used by STAS, since the circumstances of fusion and cooling would probably not have been altogether the same, and it seemed best to use silver fully purified in this respect, so that my results might be directly comparable with any obtained in the future, since this source of error once pointed out ought not to be hereafter neglected. I have therefore used as the atomic weights of silver and bromine the numbers obtained by STAS (from his experiments on silver bromide and bromate), recalculated on the assumption that the metal employed by him would have yielded 57 c.c. (reduced to 0° C., and 760 m.m.) or 82 mgrm. of oxygen per kilogramme, this being the quantity obtained by DUMAS from silver treated as nearly as possible as was in all likelihood that which STAS employed. This has the advantage of reducing the remaining error to that only which depends on the *difference* between the real amount of oxygen which was

* *Loc. cit.*

present and that assumed in such calculation, instead of leaving the whole resulting from using STAS' numbers uncorrected, my silver having had the oxygen removed, while his had not been so treated. The atomic weights adopted then for these two elements are—

$$Ag = 107{\cdot}649 \qquad Br = 79{\cdot}754$$

Calculated results.—The following are the values obtained for the atomic weight of aluminum from the different series of experiments, with the probable (mean) value resulting from each set, the difference from this mean of each individual experiment, and the probable error of the mean itself calculated in the usual way by the method of least squares:

First series.

A.

Experiment.		Diff. from mean.
I.	Al=27·029	−·011
II.	„ 27·043	+·003
III.	„ 27·055	+·015
IV.	„ 27·068	+·028
V.	„ 27·005	−·035
Mean	„ 27·040	
Probable error of mean result		±·0073

B.

Experiment.		Diff. from mean.
VI.	Al=27·114	+·018
VII.	„ 27·095	−·001
VIII.	„ 27·107	+·011
IX.	„ 27·067	−·029
X	„ 27·096	·000
Mean	„ 27·096	
Probable error of mean result		±·0054

Second series.

A.

Experiment		Diff. from mean
I.	Al=27·035	+·001
II.	„ 27·021	−·013
III.	„ 27·046	+·012
Mean	„ 27·034	
Probable error of mean result		±·0049

B.

Experiment.		Diff. from mean
IV.	Al=27·028	+·005
V.	„ 26·993	−·030
VI.	„ 27·036	+·013
VII.	„ 27·028	+·005
VIII.	„ 27·030	+·007
Mean	„ 27·023	
Probable error of mean result		±·0052

C.

Experiment		Diff. from mean.
IX.	Al=26·999	−·019
X.	„ 27·034	+·016
XI.	„ 27·021	+·003
Mean	„ 27·018	
Probable error of mean result		±·0069.

Third series.

A.			B.		
Experiment.		Diff. from mean.	Experiment.		Diff. from mean.
I.	Al=27·012	+·007	VII.	Al=26·995	+·005
II.	„ 27·005	·000	VIII.	„ 26·999	+·009
III.	„ 27·022	+·017	IX.	„ 26·977	−·013
IV.	„ 26·988	−·017			
V.	„ 27·006	+·001			
VI.	„ 26·996	−·009			
Mean	„ 27·005		Mean	„ 26·990	
Probable error of mean result		±·0033	Probable error of mean result		±·0046

In view of the gradual loss of water which, as has been shown, crystallised ammonium alum undergoes on exposure to the atmosphere, I feel that of these various sets of experiments, B of the first series is entitled to least confidence, and the considerable difference between its results and the others leads me to favour its rejection. On the other hand, I am inclined to attach most weight to series 3, A, since the method used was very simple in principle, the determination of one of the two quantities concerned was rendered very exact by the great volume occupied by the hydrogen, the comparison was made directly with the standard element in our system of atomic weights and not through the intervention of any other substance whose atomic weight must be assumed, and the agreement of the results among themselves is particularly good, as shown by the probable error of the mean being the smallest reached.

General mean of results.—The general mean from all of the thirty experiments, if all be included in the calculation, is Al=27·032, with a probable error for this mean of ±·0045.

If series 1, B, be excluded, the mean of all the remaining twenty-five experiments is Al=27·019, with a probable error of ±·0030. The third decimal having no positive value, we may take Al=27·02. If integer numbers be used for O, N, C, Na, &c., Al=27.

All the experiments which I have made are reported, except three or four in which there was manifest failure, as by accidental loss of material to a visible extent, and which were on that account not completed.

The general result adds, I trust, aluminum to the, unfortunately still limited, list of those elementary substances whose atomic weights have been determined within the limits of precision attainable with our present means of experiment.

*Bearing of final result upon "*PROUT'S *law."*

It is interesting to observe that this result also adds one to the cases already on record of the numbers representing carefully determined atomic weights approaching closely to integers, and leads to a word on the reconsideration of "PROUT'S law." The recent researches of Mr. LOCKYER, not unsupported by evidence drawn from other sources, have tended to suggest the possibility, at least, that the forms of matter which as known to us under ordinary conditions we call elements may be susceptible of progressive dissociation at enormously high temperature, and, under circumstances in which this supposed state of dissociation admits of being spectroscopically observed, some of the characteristic features in the spectrum of what is usually known to us as hydrogen become in a very remarkable degree prominent. If such dissociation may really occur, and if the atoms of hydrogen as commonly known to us form either the last term, or any term not far removed in simplicity from the last, in the progressive breaking up of other forms of matter, it is obvious that "PROUT'S law," or some modification of it, such as was many years ago suggested by DUMAS, *must* be true, the atomic weights of all the other so-called elements must be multiples of that of hydrogen, or multiples of that fraction of the hydrogen atom which may result from the dissociation of this body itself. If such fraction be very small as compared with the effect of the inevitable errors of experiment, the experimental verification or refutation of the law will prove impossible, but if it be considerable, as for instance one-half of the commonly known hydrogen atom, or one-fourth, as assumed by DUMAS, the question admits of practical examination.

Well deserved attention has for some years past been given to the labours of STAS in this direction, and his main result is no doubt properly accepted, if stated thus, that the differences between the individual determinations of each of sundry atomic weights which have been most carefully examined are distinctly less than their difference, or the difference of their mean, from the integer (or one-half or one-fourth unit which PROUT'S law would require.

But the inference which STAS himself seems disposed to draw, and which is very commonly taken as the proper conclusion from his results, namely, that PROUT'S law is disproved, or is not supported by the facts, appears much more open to dispute.

It must be remembered that the most careful work which has been done by STAS and others only proves by the close agreement of the results that *fortuitous* errors have been reduced within narrow limits. It does not prove that all sources of *constant* error have been avoided, and indeed this never can be absolutely proved, as we never can be sure that our knowledge of the substances we are dealing with is complete. DUMAS' late observations on the occlusion of oxygen by metallic silver constitute an illustration of this, some of the best of STAS' results being thereby undeniably vitiated, though probably to but a minute extent.

Of course one distinct exception to the assumed law would disprove it, if that exception were itself fully proved, but this is not the case.

As suggested by MARIGNAC and DUMAS, anyone who will impartially look at the facts can hardly escape the feeling that there must be some reason for the frequent recurrence of atomic weights differing by so little from accordance with the numbers required by the supposed law.

As the question stands at present, the following 18 atomic weights are the only ones which may be fairly considered as determined with the greatest attainable precision, or a very near approach thereto, and without dispute as to the methods employed—

*Oxygen	15·961
*Nitrogen	14·010
Chlorine	35·371†
Bromine	79·757†
Iodine	126·541†
*Sulphur	31·996†
*Potassium	39·042
*Sodium	22·987
*Lithium	7·005
Silver	107·667†
Thallium	203·655‡
*Aluminum	27·019
*Carbon	11·97
*Phosphorus	30·96
Barium	136·84
Calcium	39·90
*Magnesium	23·94
Lead	206·40

If now we discard altogether DUMAS' assumption of multiples of ·5 or ·25, and consider simply the indications afforded of PROUT'S law in its original form, we may safely take the first decimal place of each of these numbers as quite freed from the influence of *fortuitous* errors, while the second decimal is nearly so in many instances. It appears that out of the 18 numbers, 10 (those to which an asterisk is prefixed) approximate to integers within a range of variation less than one-tenth of a unit. What then is the degree of probability that this is purely accidental, as those hold who carry to the extreme the conclusions of BERZELIUS and STAS? Since there are

† STAS' numbers, uncorrected for occlusion of oxygen by silver

‡ CROOKES' number modified by taking O=15·961 instead of 15·96.

five intervals of ·1 each between any integer and the ·5 which divides it from the next higher (or lower) integer, the result is given by the expression

$$5^{-18}\left[1+\frac{\lfloor 18}{\lfloor 17}\times 4+\frac{\lfloor 18}{\lfloor 16\,\lfloor 2}\times 4^{2}+\frac{\lfloor 18}{\lfloor 15\,\lfloor 3}\times 4^{3}+\ldots\ldots\frac{\lfloor 18}{\lfloor 10\,\lfloor 8}\times 4^{8}\right]$$

and the probability in question is found to be only equal to 1 : 1097·8.

Of course this result might be easily varied by assuming other limits of precision as marking accordance with the law within the range of possible constant error, but this example seems to be based upon not unfairly assumed ground in this respect, and seems sufficiently to illustrate the point that not only is PROUT's law not as yet absolutely overturned, but that a heavy and apparently increasing weight of probability in its favour, or in favour of some modification of it, exists and demands consideration.

XXIII. *Description of some Remains of the Gigantic Land-Lizard* (Megalania prisca, Owen), *from Australia.*—Part II.

By Professor Owen, *C.B., F.R.S., &c.*

Received March 22,—Read April 15, 1880.

[Plates 34–38.]

In a former Part* the author submitted to the Royal Society evidences of the above Lacertian species, a contemporary in Australia with correspondingly large Marsupial Mammals, and which, with them, had become extinct. The remains of their cold-blooded associate, received in 1858, consisted of mutilated vertebræ. They had been imbedded in drift-deposits more or less compacted, which, when traversed by streams, had become broken up by the violence of the course to which Australian rivers are subject. During the alternate periods of drought, the river-beds are laid bare, and under these conditions the remains of *Megalania* already, and about to be, described, have been exclusively found. I have not received any specimen referable to the genus from the breccia-clefts and cave-deposits of Australia.

Although the materials for restoration of the subject of the present and former papers are incomplete, especially in regard to the limbs, I am unwilling longer to defer communicating the results of study of such portions of the skeleton as have come into my hands during the last twenty years.

The most common examples have been parts of the trunk, and among these was one entire dorsal vertebra,† of which figures are subjoined (Plate 34) of the natural size.

This bone somewhat exceeds the largest of those previously described,‡ as the subjoined dimensions indicate :—

	1858.		1880.	
	inches	lines	inches.	lines.
Length of centrum	3	3	3	6
„ non-articular lower surface of centrum	2	0	2	2
Breadth of centrum	1	11	2	0
Vertical diameter to highest part of neural arch	3	4	3	9
„ including neural spine	..		5	9

* Phil. Trans., Vol. 149, 1858, p. 43.

† Transmitted by Dr. George Bennett, F.L.S., from Darling Downs, Queensland.

‡ *Loc. cit.*, Plate 7, figs. 1–4.

The following differences, though of minor import, may be noted. The anterior (articular concave) surface of the centrum has not the "shallow transversely lengthened pit at the centre":* the two very shallow depressions at the fore part of the under surface are less marked. Better defined is the pair of tubercles above the entry of the neural canal (Plate 34, fig. 2, *n*). The neural spine (*ns*) is entire: the ridge dividing the upper surface of the neural arch is continued into the sharp anterior border of the spine; this abruptly gains breadth towards its hinder part, which is traversed by an obtuse medial rising; but this, as it descends, narrows, and is continued below the root of the spine into a sharp ridge above the exit of the neural canal, dividing there the interspace between the post-zygapophyses, as the anterior ridge (*r*) does that part of the roof of the neural arch.

The spine (*ns*) of the vertebra (ib., figs. 1 and 2), which is probably from the middle of the back, is 2 inches 6 lines in length anteriorly, 2 inches posteriorly. Here the base is broadened by a pair of obtuse ridges continued from the upper part of the post-zygapophyses (*z′*) contracting as they rise, and finally subsiding upon the broader part of the spine. The antero-posterior diameter of the spine is the same throughout: the summit is truncate and is formed by a partially coalesced epiphysis. The extreme contraction of the outlets of the neural canal, noted in the former Part,† is more striking in the present large vertebra, especially that of the anterior one (ib., fig. 2, *n*), of which the vertical diameter does not exceed 3 lines.

The chief difference presented by the dorsal vertebræ of *Moloch horridus*, which for reasons subsequently given is here contrasted with corresponding vertebræ of *Megalania prisca*, besides that of general size, is the greater relative capacity of the neural canal (ib., fig. 4).

In the sacral vertebræ of *Megalania*‡ (Plate 35, figs. 1 and 2) the neural canal (*n*) is enlarged in relation to the part of the myelon which was connected with the nerves of the hinder limbs, and I infer the possession of these in proportions at least equal to those in *Moloch* from this character. Although the vertical diameter of the anterior concavity of the centrum is reduced to 1 inch 2 lines, that of the neural canal is increased to 5 lines; and at the opposite end to 6 lines: the transverse diameter of each outlet being 11 lines. The sacral centrum gains in breadth, while losing in height: the two diameters of the anterior cup (*c*) are, respectively, 2 inches 4 lines and 1 inch 3 lines. The under surface of the centrum (ib., fig. 2) is flattened, with a feeble transverse concavity along its medial third. The length of the centrum is 2 inches 2 lines; the breadth behind the base of the "transverse process" is 2 inches 6 lines. This process (*p*), combining par- and di-apophyses, extends its base over the fore half of the coalesced centrum and neural arch. It, probably, extended outward, by the addition of a coalesced costal element, to the degree shown in the sacral vertebræ of *Moloch* (ib., fig. 5, *d*) and most other Land-Lizards; it has been broken

* Ibid., p. 44. † *Loc. cit.*, p. 45.

‡ Transmitted by F. M. Rayner, Esq., M.R.C.S., from the neighbourhood of Melbourne, Victoria, 1862.

away on both sides in the Megalanian specimen described. The post-zygapophyses are relatively small and narrow. The medial ridge upon the neural arch rises at once to contribute to the neural spine, which accordingly has greater basal breadth than in the antecedent vertebræ. This, with other characters, are repeated, in miniature, by the sacral vertebræ of *Moloch horridus.*

The caudal vertebræ of *Megalania** are represented in my present collection by a single specimen (Plate 35, figs. 3 and 4) from about the middle of the tail In it the ordinary proportions of the cup and ball are resumed, with minor flattening of the centrum: but the terminal articular surfaces are less oblique, the lower border of the cup (*c*) being more produced and the corresponding part of the ball (*b*) encroaching more upon the under surface of the centrum. The transverse process (*d*) springs from the side of the vertebra further and more distinctly from the pre-zygapophysis (*z*) than in the trunk-vertebræ: with the spinous process it is broken away, but is depressed in shape as far as preserved. A pair of hypapophyses (ib., fig. 4, *h*) rise, with an interspace of 3 lines, from the under surface of the centrum, near the ball (*b*): a smooth surface on one of them indicates the hæmal arch and spine to have been movably articulated, not anchylosed as in *Moloch* (ib., fig. 6, *hs*), to these processes. The length of the centrum of the described vertebra of *Megalania* is 2 inches 1 line; the breadth of the neural arch is 2 inches 8 lines.

The occipital segment of the skull of *Megalania* (Plate 36, figs. 1 and 2,)† shows, as in most mature Lizards, confluence of its constituent parts or "elements."

The centrum ("basioccipital," ib, fig. 1, 1) convex posteriorly, as in the following vertebræ, forms the lower half of the occipital condyle, the upper portions being contributed by the bases of the neurapophyses ("exoccipitals," ib., 1, 2). The original sutures between these and the centrum are indicated by slight linear depressions. The condyle is crescentic in shape and projects wholly behind or beyond the neural arch; the upper surface of the centrum in advance of the condyle shows a transverse excavation. The breadth of the condyle is 1 inch 9 lines; the medial depth is 9 lines.

The neural canal ("foramen magnum," ib., *n*), the side walls of which are due to the exoccipitals, (2,′ 2′), is completed in its upper third by the base of the neural spine (superoccipital, ib., 3); slight linear impressions indicate the original junctions of the latter with the exoccipitals. The outlet of the neural or cerebral canal (*n*) is subcircular, 1 inch 3 lines in diameter, but as it advances it contracts to a diameter of 1 inch, and this is encroached upon by the lateral ridges (ib., fig. 2, *n′*).

About three inches extent of the base of the skull is preserved in advance of the occipital condyle; it is formed by the coalesced basioccipital and basisphenoid. The basioccipital curves down at its mid-part and extends laterally, with a similar curve, to form the parapophyses which, beyond the outlets of the vagal nerves, coalesce with

* Collected by M. St Jean, at Gowrie, near Drayton, Darling Downs, Queensland, 1866, presented by Sir Daniel Cooper, Bart., K C.M.G.

† From the same locality as the "caudal vertebra," and received at the same date.

the diapophyses or "paroccipitals" to form strong triedral costal processes, more or less broken away, in the fossil (ib., figs. 1 and 2, 4). The lower surface of the basioccipital is transversely convex at the middle part, concave on each side.

The basisphenoid develops laterally a pair of hypapophyses in the form of low tuberosities, for the attachment of strong "recti capitis" muscles; on the left side the process shows a second pointed prominence. In advance of these are the bases of larger outstanding processes against which the pterygoids abut, in *Moloch* and other Lizards; these processes (ib., fig. 2, 5', 5') answer to the pterapophyses in the basisphenoid of Birds. At the anterior interspace of the above processes is the small surface (ib., fig. 2, 9)—here fractured—from which the presphenoidal rostrum was continued. Above the base of this process is the "sella turcica" (ib., fig. 2, 5), into which open a pair of vascular canals. The sides of the "sella" are produced into a pair of sub-compressed processes, as in *Iguana* (ib., fig. 4, 5). *Megalania* here differs, mainly, in the deeper excavation of the sella and the closer approximation of the vascular (entocarotid) canals.

The next difference is shown by the non-articular termination of the pre-zygapophyses (ib., figs. 1 and 2, 5', 5) of the occipital vertebra, and also of its spinous process (ib., *ns*), which ends obtusely and freely, like the neural spine of a trunk-vertebra.

The post-zygapophyses are those which are applied and anchylosed to the long and strong substrihedral paroccipitals (ib. 4), extending outward and backward from the sides of the centrum and neural arch of the occipital segment. The part extending from the sides of the centrum is much less than that from the neurapophyses, from which it is divided by the nerves issuing from the vagal foramina (ib., fig. 1, *v*, *v*), which pierce the base of the neural arch obliquely from within, outward, and backward, thus indicating the par- and di-apophysial constituents of the strong occipital transverse processes.

In the trunk-vertebræ of *Megalania* the neurapophyses, as they rise to form the side walls of the neural canal, develop, as has been shown (Phil. Trans., 1858, p. 44, Plate 7, fig. 4, *n*), a ridge projecting inwardly from their medial surface about half way between the floor and roof of the canal, the ridge beginning in advance of the hinder outlet. The exoccipitals repeat this neurapophysial character, but the encroaching ridges have greater basal extent and terminate obtusely, converting the anterior outlet of the occipital vertebra (Plate 36, fig 2, *n'*) from a circular to a triangular or triradiate figure. The anterior outlet of the occipital neural ("epencephalic") canal is similarly modified in *Iguana*, but with a minor prominence of the inner side-surface of the neurapophyses. Accordingly, the triradiate form of the front outlet is more marked in *Megalania*, and both vertical and transverse dimensions are relatively less than in *Iguana*, still less than in the more diminutive *Moloch* (ib. fig. 5).

The more instructive part of the skull in advance of the occipital segment, in relation to the Lacertian affinities of *Megalania*, has come to hand in the present year.

I was favoured by receiving from George Fred. Bennett, Esq., Corr. Member of the Zoological Society of London, a letter of November 27th, 1879, in which he writes :—"You will be pleased to hear of a new discovery which my father received on November 16th. I got it in King's Creek, part of Clifton Run, of which my father speaks in a letter to me, as follows :—'On examining the fossil skull you sent I considered it Reptilian, and at first sight to be a Turtle; but on further inspection there are some points which are against that opinion. There is, at present, some difficulty in solving the question. Therefore, try to get, if possible, the lower jaw and other portions of the animal, so that it may be as perfect as possible to make a drawing or cast of it before sending it to Professor Owen.'"

In reference to the parts which I have received, Mr. G. F. Bennett writes —"The whole lot was got in such a mixed way that it is hard to divide them. The letters R for 'Reptilian' and D for 'Diprotodon,' in the sketch, will give you an idea of it. They were excavated in a very hard red drift and had to be dug out with a pick: close to them was a very large jaw of *Diprotodon*. It is my intention to work out the whole of this bank. The horns were apparently in front as if they were originally in front of the eyes, but came away when digging; but I found they were not joined together to the other part of the skull."

In a letter from the father, my old and esteemed friend Dr. George Bennett, F.L.S., of December 20th, 1879, he announced the transmission of this collection, and inclosed photographs of ten of the larger portions of the supposed Reptilian skull.

They included unquestionable horn-cores and the fore part of an upper jaw, showing no trace of teeth or sockets on the alveolar border. Such correlation of horns with edentulous premaxillaries might well suggest the co-existence of a large herbivorous Mammal, which, if Marsupial, would be much nearer akin to the placental Ruminants than are the Kangaroos. But the edentulous border showed unequivocal evidences of having been encased, like that of a Chelonian, with horn, as my friend at first sight surmised: yet in a fashion more like that of the terrestrial plant-eating Tortoises* than of the fucivorous Turtles.†

On restoring the cranium as far as its transmitted fragments could be correctly juxtaposed, it manifested, in one part, not only a well-defined surface from which an apparently autogenous horn-core, as in the Giraffe, had become detached, but also pairs of exogenous ones like those of the Ox. The longest of these extended from the upper and side borders of the hinder portion of the cranial specimen, but evidently anterior, as in the Bison, to the occipital ridge. The surface, seemingly, for the sutural attachment of a horn core was on the upper part of the nasal bone, symmetrical in shape, crossing the mid-line like the horn of a Rhinoceros.

The breadth of this many-horned skull from tip to tip of the pair of horn-cores (Plate 37, fig. 1, *b*, *b'*) is 1 foot $10\frac{1}{2}$ inches: the length of the recomposed extent,

* Cuvier, 'Ossemens Fossiles,' 4to., tom v, pt. 2, plate 11, fig 19, *a*, *b*

† *Tom. cit*, plate 11, fig. 3, *e*, *e*.

including the natural anterior end of the skull (Plate 38, fig. 1) is 1 foot; but the occiput, here, is wanting.

The length of each of the horn-cores (b, b') is 5 inches: the circumference of the base of the core is 8¾ inches: this is of a full oval shape, 3 inches across antero-posteriorly, 2 inches vertically. These cores are conical, nearly straight; the upper contour is slightly concave lengthwise near the base and also at the apex, but runs straight between them; the lower contour is very slightly but uniformly convex. It may be inferred that the horny sheath (b') of the core showed a more definite, though feeble upward curve. Both cores extend almost horizontally outward, transversely to the long axis of the skull. Their surface is impressed by small vascular holes and channels.

A second shorter core (Plates 37 and 38, fig. 1, e), projecting from the fore part of the base of the preceding, might be reckoned a branch thereof, as if foreshowing the type of the brow-antlered weapon of the deer.

The breadth of the cranium between the hind part of the bases of the ("supra-temporal") horns (b, b') is 13½ inches. This dimension rather rapidly contracts towards the orbits (o, o), reducing the breadth between the fore part of the post-orbital tuberosities ($_{12}$) to 8½ inches. Thence the skull narrows to the single median nostril (Plate 37, fig. 1, ol), between the outer walls of which it is reduced to 4 inches in breadth, the vertical diameter being about the same. The breadth of the preserved maxillo-premaxillary arch (ib. ib., $_{21}$, $_{22}$) is 6⅓ inches.

The base of the small horn-core (e) is subcircular, about 2 inches in diameter; its length is 1½ inch; it ends more obtusely than the core (b), of which it seems to be a branch, and projects obliquely downward and outward.

Somewhat in advance of this rises a third larger core, or tuberosity (Plates 37 and 38, fig. 1, c), projecting outward from the upper and lateral border of the cranium, overhanging the broad vertical zygomatic plate ($_{26}$, $_{27}$). The base of the core (c) measures about 3 inches from behind forward, and rather less transversely; it rises about an inch above the cranial level.

A lower tuberosity (ib. ib., f) with a broader base rises from the upper surface of the cranium, behind the last described and nearer the mid-line: its summit, rugged as in the others, is more transversely extended or ridge-like; the horn which it probably supported would be characterised by its transversely broad base.

Each of the cores (c, e, f), like that marked b, is one of a pair, of which the first (b) and third (f) are preserved on the left as well as on the right side of the cranium; the core (c) has been broken away with its supporting part of the cranial plate on the left side.

The rugose surface (ib. ib., d) upon the nasal region indicates the sutural attachment, or fractured base, of a horn-core. It is single, symmetrical, transversely ellipsoid in shape, 2½ inches across, and 1 inch in the skull's axis. It is situated 2¾ inches behind the fore end of the skull.

On the right side of the skull, $4\frac{1}{2}$ inches of the outer wall (ib., *oc*) extends behind the supra-temporal horn-core (*b*). The broken edge of this portion of skull gives $\frac{1}{2}$ inch thickness of rather compact bone, increasing upwards to $1\frac{1}{4}$ inch, with a finely cancellous texture occupying the middle two-thirds. The hind broken margin of the upper wall of the skull thins off to 2 or 3 lines From the under surface and mid-line of this border a septum (Plate 37, fig. 1, 7′) descends for about the extent of an inch, answering to that which similarly descends from the line of the sagittal suture to articulate, in *Chelone*, with the subjacent occipital spine. On each side of this indication by the medial septum of capacious temporal fossæ the inner smooth compact surface of the bone is largely and deeply excavated.

The nasal aperture (Plates 37 and 38, fig. 1, *ol*) is overhung by the coalesced nasal bones (ib., 15) which project about $1\frac{1}{2}$ inch in advance of the premaxillary (ib. ib., 22). The aperture is $2\frac{3}{4}$ inches in breadth, 1 inch 2 lines in height, partially bisected by a septal process (*n*) of the premaxillary, which rises towards, but does not meet, the corresponding thinner septal process from the under part of the nasal.

The premaxillary (ib., 22) is continued from the nostril downward, with a slight backward curve, about 2 inches beneath the olfactory cavity, the floor of which curves downward at the outlet, with little mark of boundary, upon the fore surface of the premaxillary.

All sutures are obliterated: at least I have not been able to determine any, satisfactorily, in the portions of the cranium transmitted.

A shallow channel (Plate 38, figs. 1 and 2, *on*), $\frac{1}{2}$ inch broad, is continued from the nostril to the orbit, dividing, as it seems, the nasal bone (15) from the maxillo-premaxillary (21, 22) on each side of the outer surface of the skull, the channels converging as they advance to the sides of the nostril.

The lower border of the maxillo-premaxillary (ib., fig. 3, 21, 22) is subtrenchant, convex forward; the breadth of the preserved part across the ends of the semicircle, beneath the orbits, is $6\frac{1}{2}$ inches. An irregular notch at the middle of the fore part of the edentulous border (ib., fig. 2, 22) seems due to accident. The sides as well as fore part of this evidence of the upper jaw curve downward and inward.

The upper surface of the nasal (15), supporting the surface for the horn-core (*d*), is broad, flattened, but laterally bends down somewhat abruptly to form the antorbital side-wall of the nasal cavity. The platform supporting the front horn (*d*) shows great thickness and strength.

The extent of skull between the nostril and orbit is 2 inches. The preserved border of the orbit (Plates 37 and 38, fig. 1, *o*) is sub-semicircular in shape. Its upper boundary extends from the orbito-nasal groove obliquely upward and backward for the extent of 2 inches; the hind boundary then bends abruptly downward and rather backward; the lower boundary is broken away on both sides of the skull; about half of the orbital floor is preserved on the left side; rather more of the roof (Plate 38, fig. 3, *o′*, *o′*) is preserved on both sides. There is a feeble prominence at the middle

of the broad or thick superorbital boundary, which may indicate the share which the frontal (11) contributed to form, with the pre- and post-frontals (12), that part of the orbital frame. The latter cranial element is indicated by the subangular prominence (ib., 12) at the hind part of the boundary. The dimensions of the orbit are 2 inches in vertical, $2\frac{1}{4}$ inches in longitudinal, diameter. These relatively small cavities open upon the anterior third part of the skull—perhaps on the anterior fourth were the skull entire.

The side-wall of the skull (Plate 38, fig. 1, 26, 27), behind the orbit extending back beneath the horn-core (*e*), is vertical and (apparently has been) entire, descending from the base of the horn-core (*c*) for an extent of 5 inches. Beneath the core (*e*) is a low prominence; behind this part, and below the base of the core (*b*), is a natural vacuity (ib., fig. 1, *t*), of a vertically oval form, 1 inch 11 lines in that diameter, and apparently 1 inch 6 lines from before backward. This vacuity leads to the temporal fossa, which is entirely roofed over by the broad arched external cranial platform, developing and supporting the lateral ("supra-temporal," *b*, *c*) and superior ("supra-parietal," *f*) pairs of horn-cores.

The side-wall descends below the vertical temporal opening for an extent of 2 inches 10 lines. The outer surface of this part is slightly convex, and sculptured by some vascular channels; it is bounded behind by a shallow groove, 9 lines across, extending from the temporal opening obliquely downward and backward. The groove is bounded behind by a small rugged mamilloid process (8).

The above-described "side-wall" holds the place of a malo-squamosal zygoma (26, 27); the hinder terminal tuberosity (ib., 8) I regard as part of the mastoid.

On the left side of the skull a bony plate is continued from part of the inner circumference of the vertical temporal aperture for some distance transversely mesiad. The hinder part of this production is much thicker than the fore part, and extends further before ending in a broken surface: it was, probably, continuous with the proper parietes of the brain-case, of which, however, no trace remains.

The major part of the extensive horn-supporting plates seems to correspond with those parts of the parietals and mastoids in *Chelone*, which make the vaulted roof of the temporal fossæ.

Of the tympanics, or supporting bones of the tympanic membrane in *Chelonia*, and of the articular surfaces for those bones, there is no preserved trace.

Turning to the under surface of the present cranial specimen, the fore boundary of each "palatonaris" (Plate 38, fig. 3, *pn*) is preserved. The roof of the mouth extends forward 2 inches from these boundaries. On this palatal part of a broad upper jaw, formed, as it seems, by the coalesced premaxillary and maxillaries, projects the trenchant ridge (ib. ib., *r*), extending nearly, but not quite, parallel with the outer margin (ib. ib., 21, 22); it is somewhat sharper than that margin, from which it is divided by a pretty deep regular channel, widening from 6 lines in breadth at the mid-line of the mouth to 9 lines at its outer, best preserved, end. Behind the second ridge

is a third lower but equally trenchant one (ib. ib., *s*), with a smooth canaliculate interspace of 6 lines broad between it and the second ridge, with which it runs parallel and is nearly co-extensive.

From near the lateral and posterior parts of the palatal portions of the skull the smooth arched roofs of both orbits (ib. ib., *o′*, *o′*) are preserved for an extent of from 2 to 3 inches.

The shape and superficies of the curved ridges and channels on the prepalatine part of the upper jaw indicate that such part was sheathed with horn in the living *Megalania*. This Chelonian character is associated, as I have remarked, with that of the osseous expanse over-roofing wide temporal vacuities. But the chief affinities, vertebral as well as cranial, of *Megalania*, are with the Lacertians, and more especially, as I have next to show, with a Lizard still living in and peculiar to the Australian continent.

In the existing *Lacertilia* there are four known genera with horned species. *Ceratophora, Phrynosoma, Metopoceros,* and *Moloch.* In the first genus (*hab.* Ceylon) the so-called horn is single, supra-nasal, elongate, flexible, little different in texture from the common integument. In *Ceratophora Stodarti* (Plate 37, fig. 6) it is sub-compressed, pointed, of moderate length; in *C. aspera* (ib., fig. 7) it resembles rather a short proboscis than a horn. In *Phrynosoma regale* (*hab.* California) a semicircle of antero-posteriorly, sub-compressed, broad, corneous spines (ib., fig. 8) crowns, as it were, the occiput; there are some corneous papillæ in other parts of the head. In the Iguanian genus *Metopoceros* (*hab.* S. America) the species *M. cornutus* carries a single symmetrical short horn upon the nasal region.

Only in the small Australian Lizard* (*Moloch horridus*, GRAY†) have I found a head resembling in its proportionate breadth and shortness that of *Megalania*, with the following correspondences in the cranial armature, seven of the horns (Plate 37, fig. 2, and fig. 9, *b*, *c*, *d*, *e*) answering to those similarly marked in fig. 1, ib.

The horns of the pair (*b*), which are the longest in *Moloch* as in *Megalania*, are also the largest and widest apart; they spring from the sides of the upper part of the head near to but in advance of the occiput: they are the "supra-temporal horns." A shorter horn (ib., fig. 2, *e*) projects close to the fore part of the base of (*b*), and seems a repetition of the horn marked (*e*) in *Megalania*. The horns of the pair placed nearer together and springing from the upper part of the head (ib., fig. 2, *c*) correspond to the "post-orbital pair" (fig. 1, *c*, *c*) in *Megalania*; but the large proportional size of the orbits in *Moloch* makes them more approximate. The most advanced horn (*d*) on the upper part of the head is single and symmetrical in *Moloch* and repeats the

* Discovered by JOHN GOULD, F.R.S., in the Swan River district, and exhibited by him as a "spiny Lizard allied to the Agamas," at the meeting of the Zoological Society, August 5th, 1840 Proceedings, p 94, 8vo., 1840.

† "Descriptions of some new species and genera of Reptiles from Western Australia discovered by JOHN GOULD, Esq" 'Annals and Magazine of Natural History,' vol vii, 1841, p 88

"supra-nasal horn" in *Megalania* (Plate 38, fig. 1, *d*). The larger horns in *Moloch* (Plate 37, fig. 2, *b*) curve obliquely outward and backward; the "post-orbital" (*c*) and "nasal" (*d*) horns are vertical: both are short. Moreover, in *Moloch* a pair of horns of intermediate size, but broad basally in proportion to their height (ib., fig. 2, *a*), stand upright, their bases touching each other above the occiput. There is not such good ground for homologising these with the "supra-parietal pair" (ib., fig. 1, *f*, *f*), but the extension in breadth of the summit of these horn-supporters in *Megalania* is notable in relation to this comparison.

The occipital segment of the skull had not come to hand at the date of Mr. G. F. BENNETT's discovery and transmission of the parts above described; but the previous acquisition of that cranial vertebra (Plate 36, figs. 1 and 2), by M. ST. JEAN, of another *Megalania*, in a different locality, has enabled me to show, in the free tuberous termination of the superoccipital spine (ib., *ns*) and the restricted attachment by confluence in other elements with the cranial segment in advance, characters which are more closely repeated in the skull of *Moloch* (ib., fig. 5) than in other Lacertian subjects of comparison.

Moreover, in the skull of *Moloch* (Plate 37, figs. 3, 4, 5) a single edentulous premaxillary (22) is sheathed with horn. It is relatively smaller than the similarly edentulous upper jaw-bone of *Megalania*, and the retained sutures in the small horned Lizard show the articulation at each side-end of (22) with a maxillary (21). The premaxillary sends up a short medial process which partially divides the single external nostril (ib., fig. 5, *ol*), but does not reach the nasal (15). This bone forms, as in *Megalania*, a broad, mainly horizontal platform, uniting behind with a similarly broad mid-frontal, and laterally with the nasal process of each maxillary (ib., fig. 5, 21) and with a portion of each pre-frontal (14).

Each maxillary in *Moloch* supports a single series of minute teeth, acrodont in attachment, slightly increasing in size as they recede in position. These denticles are close set, 20 in number in each maxillary. The malar is broad, ascends obliquely backward to join the post-frontal, and, by a short slender process, combines with a short and broad squamosal to bound the temporal fossa (ib., *t*). A small mastoid and the posteriorly-produced angles of the parietal form the joint for a broad tympanic (ib., fig. 4, 28).

There is a small transverse vacuity between the parietal and frontal representing the "foramen parietale;" a pair of similar "fontanelles" open between the parietal and super-occipital, the division being partially made by the anteriorly-produced tuberous occipital spine (fig. 3, 3). The palatonares (ib., fig. 4, *n*) are small and anterior, divided by transversely-extended palatines from the larger pterygo-maxillary vacuities (*y*) behind. The parietal develops a pair of short conical processes or "cores" for the support of the vertical horns (fig. 5, *e*, c). There is no such process for the single nasal horn (*a*).

In the skeleton of *Moloch horridus* (ib., fig. 9) there are 21 vertebræ between the

skull and sacrum; of these, 14 support movable ribs, leaving four "cervical" and three "lumbar." The two sacral vertebræ have sub-depressed centrums, and send outward long and strong costal processes (Plate 35, fig. 5), which converge to abut against the vertical ilium. The tail includes about 20 vertebræ, many of which (ib., fig. 6) support a hæmal arch and spine (*hs*) upon a pair of hypapophyses near the hind ball of the centrum. The costal vertebræ (Plate 34, figs. 3 and 4) are miniature repetitions of those in *Megalania:* the anterior cup and posterior ball show a similar shape and obliquity of position The neural arch has coalesced with the centrum, which is relatively to the spine rather longer. The rib articulates with a single tubercle beneath the pre-zygapophysis. Of the ribs, the fifth to the ninth inclusive are connected by progressively lengthening hæmapophyses to the margins of a broad sternum. Both fore and hind limbs are pentadactyle and unguiculate. The third and fourth digits are longest; the hind foot is rather longer and narrower than the fore foot, but both show the short Agamian proportions.

A fossil fragment of flat bone, with a moderately convex border roughened as for the attachment of cartilage, 9 inches in breadth and 2 inches in thickness at the narrower fractured end, best corresponds with the expanded end of the scapula supporting the gristly superscapula in *Moloch.* This massive portion of bone was transmitted, with vertebræ of *Megalania*, from the neighbourhood of Melbourne by F. M. Raynal, Esq., in 1862.

In the Supplement, No. IX., of a 'Monograph on the Fossil Reptilia of the Wealden and Purbeck Formations,'* I described and figured certain fossils from the 'feather-bed' sub-division of the latter locality. From their number and the association of these fossils, called "granicones,"† with unquestionable remains of Lacertians, I conceived that they were, most probably, the osseous supports of horny developments of the integument, affording the extinct *Nuthetes* a defensive armour, like that of the *Moloch horridus;* and I alluded to the association of Marsupial Mammals with such fossils as supporting the interpretation suggested by the smaller Lizard, now living at the Antipodes in like Mammalian association. The dermal interpretation of the osseous "granicones" was further supported by their intimate texture, the evidences of which were submitted to the Royal Microscopical Society of London.‡

In the investigation of the structure of the horns and spines of *Moloch horridus* it was found that the density of the supporting cones of fibrous corium was not augmented by bone-deposits, but indications of such decussatory fibrous structure in the ossified cones were plain.

In the huge extinct horned Lizard of Australia the horn-cores, as we have seen, are ossified, and the texture of the bone, as revealed by the microscope in thin transparent

* Volume of the Palæontographical Society, issued 1879, 4to.

† *Op. cit.*, p. 15, Plate 11, figs. 17–21.

‡ Journal of the Royal Microscopical Society, 8vo, vol. i, 1878, p 233, Plates 12, 13.

sections, closely agrees with that of the granicones or assumed cores of the horns and spines of the extinct Lizard associated with the Marsupials at the Upper Oolitic or Wealden period in our own island.

In other, older, and larger extinct Saurians * the osseous supports of such horns and spines are likewise, as in *Megalania*, the sole evidences remaining.

It is interesting here to note the continuance of multiplied pairs of cranial horns in certain extinct Mammals of the earlier tertiary periods: as, *e.g.*, the four-horned Sivathere and Bramathere of the Sivalik miocene. In the *Dinocerata* (MARSH †) of the Rocky Mountain miocene, the number of such seeming cores comes still nearer to that in *Megalania*. True it is that objections to the term "horns" applied to certain elevations of the outer table of the skull in that extinct family, might similarly affect those marked *f* in the skull of *Megalania*, but the weapons which feebler cranial indications sustain in *Moloch* have weighed with me in the foregoing descriptions.

What, it may be asked, were the habits of life of this hugh Australian reptile of diabolic aspect? With regard to its small existing representative, the name *Moloch horridus* is expressive of the emotions excited by its physiognomy rather than indicative of its zoological characters, and the nature of such emotion may be judged by the nomenclator's admission that "the external appearance of this Lizard is the most ferocious of any that I know." ‡

It is nevertheless a poor harmless, timorous, little Lizard; a contemporary it may have been with *Megalania*, and indebted for its continued existence, as a species, to a dwarfishness favouring concealment, and to such defence as its tegumentary spines may offer against the small existing and contemporaneous predatory enemies. The reptilian *Megalania*, from present dental evidence, seems to have been phytiphagous, and accordingly, like many herbivorous Mammals, it was provided with defensive weapons. These would be as available against the attacks of *Thylacoleo* as the Buffalos' horns are against those of the South African Lion. But the time at length arrived when a more fell destroyer then either the marsupial or placental four-footed Carnivore came upon the stage. Then, I conclude, drew nigh the date of extirpation of every large animal that afforded meat to the Australian, so called, "Aborigine."

Hence the Naturalists' knowledge of the huge species, so extirpated, rests upon reconstructions based on comparisons of their fossil remains.

* *Hylæosaurus* (dorsal spines), MANTELL: Phil. Trans., 1841, p. 131

Scelidosaurus (dorsal, lateral, subcaudal spines), OWEN: 'Monograph,' in 4to. vol. of Palæontographical Society, issued 1862, p. 20, plates 1–9

Iguanodon (carpal spines), OWEN: 'Monograph,' in 4to. vol. of Palæontographical Society, issued 1872, p. 6, plates 1 and 2

Stegosaurus (numerous spines—position uncertain), MARSH: 'American Journal of Science and Arts,' vol. xix., 1880, p. 258

† MARSH: 'American Journal of Science and Arts,' vol. v., 1873, p. 486.

‡ GRAY, *loc. cit.*, p. 88.

Addendum.

(Added October 15, 1880.)

From the subjects of Part I. (1858) the inference was that a Land-Lizard of unusual size had existed in Australia; suggesting further quest of such character among existing Lizards, with the result that the largest known species was Australian. Any doubt that lingered was as to the validity of separating generically *Megalania* from *Hydrosaurus*. It was possible that the discovery of the fossil skull might require a generic synonymy with the existing carnivorous species; and it seemed very probable that, in habits, the extinct might agree with the large existing Lizard.

As years elapsed with successive acquisitions of other parts of the great extinct Lizard, and especially the reception, in 1866, of the occipital vertebra, these fossils led to the detection of such differences in the corresponding parts of *Hydrosaurus giganteus* as to confirm the original inference of generic distinction. But other portions of the skull were wanting to deal profitably with this advance of insight. Accordingly, my instructions and desires were pressed importunately on friendly collectors and transmitters of Australian fossils; and, in 1879, met with the return which has enabled the above contribution of Part II.

Description of the Plates.

PLATE 34.

Fig. 1. Side view of a dorsal vertebra. *Megalania prisca.*
Fig. 2. Front view of the same. Ib.
Fig. 3. Side view of a dorsal vertebra. *Moloch horridus.*
Fig. 4. Front view of the same. Ib.

All the figures are of the natural size.

PLATE 35.

Fig. 1. Back view of a sacral vertebra. *Megalania prisca.*
Fig. 2. Under view of the same. Ib.
Fig. 3. Front view of a caudal vertebra. *Megalania prisca.*
Fig. 4. Side view of the same. Ib.
Fig. 5. Front view of a sacral vertebra. *Moloch horridus.*
Fig. 6. Side view of a caudal vertebra with the hæmal spine. Ib.

All the figures are of the natural size.

PLATE 36.

Fig. 1. Back view of the occipital vertebra. *Megalania prisca.*
Fig. 2. Front view of the same. Ib.
Fig. 3. Front view of the occipital vertebra. *Iguana tuberculata.*
Fig. 4. Back view of same. Ib.
Fig. 5. Back view of the occipital vertebra. *Moloch horridus.*

All the figures are of the natural size.

PLATE 37.

Fig. 1. Oblique upper view of the skull (wanting under jaw); nearly half the natural size. *Megalania prisca.*
Fig. 2. Oblique upper view of the head. *Moloch horridus.*
Fig. 3. Upper view of the skull. Ib.
Fig. 4. Under view of the skull. Ib.
Fig. 5. Front view of the skull (wanting under jaw). Ib.
Fig. 6. Side view of the head. *Ceratophora stodarti.*
Fig. 7. Side view of the head. *Ceratophora aspera.*
Fig. 8. Front view of the head. *Phrynosoma regale.*
Fig. 9. Skeleton, with outline of body. *Moloch horridus.*

All the figures, save fig. 1, are of the natural size.

PLATE 38.

Fig. 1. Side view of the skull (wanting the occipital segment and under jaw); one-third the natural size. *Megalania prisca.*
Fig. 2. Front view of fore end of skull; half the natural size. Ib.
Fig. 3 Under view of the same; half the natural size. Ib.

XXIV. *On the Ova of the* Echidna Hystrix.

By Professor OWEN, *C.B., F.R.S., &c.*

Received April 26,—Read May 13, 1880.

[PLATE 39.]

TOWARDS a knowledge of the generative economy of the Spiny Monotreme (*Echidna Hystrix*, CUV.) the recorded steps are the following:—

The fact of its possessing mammary but teat-less glands, as in the *Ornithorhynchus*;* that these glands acquired large and functional development concomitant with ovarian indications of recent gestation;† that the lacteal areola became lodged in a tegumentary depression or *quasi* pouch, capable of receiving the head and fore limbs of the young,‡ at least when this was not more than 1 inch 10 lines in total length.

The female *Echidna* and her young, the subjects of the paper of 1865, were taken in Lolac Forest, Victoria, Australia, on the 12th August, 1864. Guided thereby, I noted, in correspondence with friends in localities frequented by the Echidnæ, the period when females in the impregnated state might be obtained, with instructions as to the parts to be preserved and transmitted, in alcohol, for examination; noting, also, the chief facts which remained to be determined§ in reference to the subject of the present communication. Among such friendly correspondents I have the good fortune to include GEORGE FREDERIC BENNETT, Esq., Corresponding Member of the Zoological Society of London, resident at a locality, Toowoomba, in Queensland, where individuals of the *Echidna Hystrix* were to be had. In a letter of September 23rd, 1878, Mr. BENNETT writes:—"You will have received, ere this reaches you, specimens of probably impregnated *Echidna* got on various dates—July 18th, 27th, and August 9th."

The correspondent's father, my friend Dr. BENNETT, F.L.S., being in London when these specimens had arrived, I dissected them in his presence, but found not any ovum in either uterus.‖ The ovaria showed one or more enlarged ovisacs.

As the season of generation might vary within certain limits in different localities, I urged the prosecution of the quest through the months of August and September.

* Phil. Trans., 1832, p. 537, Plate 17, figs. 2 and 3.

† *Op. cit.*, 1865, p. 674, Plate 40, figs. 1–5.

‡ Ibid., p. 675.

§ Ibid., p. 677, Plate 40, figs. 6–10. Ibid, pp. 672, 682.

‖ See the figure of the female generative organs of *Echidna Hystrix*, in Phil Trans, 1865, Plate 41, fig 1.

In a letter from Mr. G. F. BENNETT, dated Toowoomba, January 14th, 1879, he writes:—"I was not able to get any more specimens this last season, but I hope to begin where I left off last year, and continue to the time advised in the letter of my father's—that is, to the middle of September; and then I hope I shall be able to give you material to decide this important question."

In February last I was favoured by receiving from Dr. BENNETT, who had returned to Sydney, New South Wales, a letter of the date December 26th, 1879, in which I had the pleasure to read:—"I have now sent you a case containing a tin of preparations in spirit of the uteri of *Echidna*. I selected those I thought would interest you, but I fear none of them yet solve the problem. One (not sent) I have ventured to open—distended uterus of one captured in August—and it contained a small semi-opaque ovum. Those sent were captured from August 30th to October 10th. One, taken September 24th, was not impregnated, but a young one was found in the pouch, and another on the 29th September."

I welcomed this satisfactory confirmation of the marsupial structure described in the paper of March 2nd, 1865,* and was gratified shortly after by receiving the "tin" of specimens notified as despatched by my friendly correspondent.

The specimens, four in number, of the hinder half of the trunk and hind limbs, with the generative organs entire and *in situ*, were in a good state of preservation; and my first leisure was devoted to their examination. In two of these the uteri were unimpregnated. In a third specimen, from a female killed August 30th, 1879, three ova were lodged in the deep folds of the thick and soft inner membrane or tissue of the left uterus (Plate 39, fig. 1, c'). There was no ovum in the right uterus (ibid., fig. 1, d, c).

The ova were of a spherical form, and of different sizes; the smallest had a diameter of $2\frac{1}{2}$ millims., the next in size of 4 millims., the largest of 6 millims.

The smallest ovum was situated nearest the vaginal end of the uterus, the next in size was nearer the Fallopian end, the largest held an intermediate position. Each was lodged in a smooth depression of the soft and thick inner uterine coat, and some threads of mucus extended over the larger ovum. These were shown by microscopical investigation not to be vessels, but filamentary portions of uterine secretion (fig. 2, a a), coagulated probably by the preserving fluid. A magnified view of the ovum, as seen undisturbed in its nidus, c', is given in fig. 2.

In the fourth specimen, from a female killed on the 14th September, 1879, there was no ovum in the left uterus (ibid., fig. 3, c', d'); but in the right uterus (ibid., fig. 3, c, d) was an ovum of the size of the largest of the three in that of the date August 30th. It is shown *in situ*, with half of the smooth cell from which it was dislodged in reflecting the portion of the uterine wall, c, including the ovum with the other half of the cell. This was situated near the Fallopian end of the uterus. The parts being dissected while under colourless dilute alcohol, a slight touch of a camel's

* Phil. Trans., 1865, p. 671, Plate 39.

hair pencil rolled the ovum out of its cell; there was no organic adhesion between them.

In Plate 39, fig. 4, the ovum is shown, moderately magnified. The outer membrane (hyalinion or "zona pellucida") closely resembled that of the similar-sized ovum of the *Ornithorhynchus paradoxus*.* Under a magnifying power of 120, hyaline corpuscles with interspaces, as shown in fig. 6, pervaded the homogeneous part of the tunic. The space separating it from the embryonal or vitelline mass is shown at the rent of the hyalinion, at *h*, fig. 4. No trace of spermatozoa could be detected in the fluid occupying the interspace.

The yolk or germ-mass has a delicate smooth covering closely attached to the subjacent mass. On picking off with a fine needle-point part of this investment (fig. 5, *v*), the granular yolk-substance adhered to its inner surface. The chief and interesting feature of this yolk-mass was the linear fissure, *g*, extending along about one-third of the periphery, and penetrating a short way into the vitelline or germinal substance. This fissure may be interpreted as the commencement of the primary fission of the mass, and as indicative of a stage in development prior or preparatory to the complete fission observed by BARRY† and BISCHOFF‡ in the smaller uterine ovum of the rabbit.

Not any trace of embryonal structure was discernible in or near this fissure, or in any other part of the germ-mass. One portion of this mass presented a darker tint than the rest, but the structure was uniformly and minutely granular; the coagulating influence of the alcoholic menstruum had given increased firmness to the mass.

The results of the foregoing investigations may be summed up as demonstrating the close resemblance in *Echidna* to *Ornithorhynchus* of the uterine ovum in structure, as in augmentation of size prior to embryonal development; the latter character being exemplified in a greater degree in the present (*Echidna*) series by the smaller size of the ovum, the least of the three, in the left uterus (Plate 39, fig. 1). Concomitantly with the more equal development of the right and left female organs in *Echidna*, as compared with *Ornithorhynchus*,§ is the evidence of an ovipont by the right ovarium shown by the reception of the impregnated ovum in the uterus of that side (ib., fig. 3).

Finally, the additional evidence of the viviparity of the Monotremes in the commencement of the fission of the germ-mass (fig. 5, *y*).

* This "membrane (Plate 25, fig. 6, *a*) offered a moderate degree of resistance when torn open, and yielded equally in every direction when separated from the yolk, the rent margins curling inward like the coat of an hydatid; it was of a dull greyish colour, slightly transparent."—Phil. Trans., 1834, p. 561.

† Phil. Trans., 1839, pp 320–332.

‡ 'Entwickelungsgeschichte der Kaninchen-Eies,' 4to., 1842.

§ Phil. Trans., 1834, Plate 25, fig. 2; Phil. Trans, 1865, Plate 41, fig. 1.

Description of the Plate.

Fig. 1. Female organs of *Echidna Hystrix;* both uteri laid open, and three ova exposed *in situ* in the left uterus. Natural size.

Fig. 2. The largest of the three ova in the cell of the lining uterine substance, with filamentary portions of secreted matter passing across the ovum. Magnified.

Fig. 3. Female organs of *Echidna Hystrix;* both uteri laid open, and one ovum shown in the right uterus. Natural size.

Fig. 4. The ovum removed, with the hyalinion, *h*, partially torn to expose the yolk or germ-mass and the interspace between it and the outer tunic. Magnified.

Fig. 5. The germ-mass of the same ovum, showing the fissure *y*; the hyalinion, *h*, being reflected. Magnified.

Fig. 6. A portion of the hyalinion. Highly magnified.

XXV. *On the Determination of the Constants of the Cup Anemometer by Experiments with a Whirling Machine*—Part II.

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Received June 13—Read June 17, 1880.

(55.)* In the preceding Part (Philosophical Transactions for 1878, p. 777) I gave the results obtained by anemometers attached to a whirling machine, which substitutes motion through the air for real wind. If the air were quiescent this method would be nearly unexceptionable; but the whirling gives the air a vorticose motion for which it is impossible to make an exact allowance, and therefore some uncertainty affects these results. In the conclusion of that paper I expressed an opinion that greater certainty might be obtained by comparing two anemometers, similar and equal in every respect except friction; and stated that I would endeavour to carry this into effect. I propose now to give an account of my attempt to do so.

(56.) The instruments used, and their arrangement, are described in paragraph (51). The situation in which they are placed would be a good one but for the dome of the west equatorial, which in some points of the wind may interfere with its full action on one or the other of the instruments.

The diameter is 13′·6; the height of its summit above the platform is 15′·75; that of the arms of the instruments being 16′. The horizontal distance of its centre from the Kew instrument (K)=21′·5, bearing from it S.S.E., 2° S. The distance from the experimental one (E) is 23′, and its bearing S W. b. S.

The distance between K and E=22′. Of course when the wind is S.S.E., K will be less acted on than E, and *vice versâ*, but probably the difference will be much less than that caused by fluctuations of the wind itself. When the wind is E. or W. the eddies caused by the windward anemometers may perhaps reach the leeward, but not, I think, to any great extent.

(57.) The chronograph record of each experiment was at first entered in groups during which v, the velocity of K, was nearly uniform; and A, the number of turns made by each instrument, was an integer. The length of the chronograph helix gives the time; it is measured in eighths of an inch (as the Observatory possesses a scale of eightieths) and when divided by the length of a second on the same scale, we have the number of seconds. As the chronograph in its present situation is exposed to considerable variations of temperature, its rate is not as regular as it was at Rathmines, but the second-space was determined each day of observations. The average in

* For facility of reference the numeration of the paragraphs and tables is in sequence to that of Part I.

winter is 1·665; and the times so deduced are certain to less than $0^{s}·1$. Latterly the time was noted by a watch.

(58.) It was soon found that the method proposed in paragraph (52) is not available, for the wind is never uniform long enough to make two successive experiments fairly comparable. It was therefore necessary to use that of paragraph (53). Assuming such values of α, x, and y, the constants of equation III. as will give V very nearly equal to V′ (the accented letters belong to E), we may correct them so that the mean V′−V may vanish. This assumes first, that however the wind may vary in the course of an observation from one instrument to another, yet if the time be sufficient it comes to each of them with an equal amount; its deficiency at one part of the time being made up by its excess at another; and secondly, that the V computed for a mean value of v will be its own mean value.

(59.) As to the first of these assumptions, I have come to the conclusion that if an observation lasts for nine or ten minutes, the average action of the wind on the two instruments will be nearly equal, though during portions of the time it may vary very much. This may be illustrated by the following table, which contains a set of v and v' taken with the normal frictions at K and E, which are 13·5 and 23·2; these were taken September 17, 1878, under unfavourable circumstances, for the wind was S.W. The v and v' ought to be nearly equal, for the difference of the friction will only diminish v by 0·24.

TABLE XX.

No.	Time.	v.	v'.	$v-v'$.	No.	Time.	v.	v'.	$v-v'$.	No.	Time.	v.	v'.	$v-v'$.
	s.					s.					s.			
1	15·1	6·522	4·660	1 869	22	5 2	6·441	6·441	0 000	43	18·4	5·349	6·876	−1·527
2	28·0	5·254	5 254	0·000	23	10·3	5·475	5·475	0·000	44	29·4	7 175	8 400	−1 225
3	25·5	7·156	6·620	0·536	24	9·2	6·132	4 599	1·533	45	50·3	7·540	6·143	1·397
4	11·5	6·084	3 650	2 834	25	17·5	8·012	6·409	1·603	46	32·4	6·077	5·208	0 869
5	11 3	7 638	6·236	1·402	26	22·0	5 702	4 474	1·228	47	9 2	4·589	5·842	−1 253
6	21·1	5 313	5·313	0 000	27	9 6	5 882	5·882	0·000	48	16·4	5 989	5 135	0 854
7	13·5	4·174	2·086	2 088	28	32 4	7 791	7·353	0·438	49	11·8	7 142	5 959	1 183
8	8.5	6 624	6·624	0 000	29	10·8	6·482	6·482	0·000	50	19·3	5 082	4 841	0·241
9	12 1	4 642	3 481	1·161	30	12·1	4 595	5 795	−1 200	51	45·3	6 512	6 512	0 000
10	28 7	2 934	4·995	0·939	31	15 2	3·697	4 622	−0·925	52	30 2	5·378	3 222	2 148
11	10·9	6·458	5 162	1·296	32	35·4	4·637	3·174	1·462	53	14·0	3 193	3 017	0·176
12	13·2	4·256	5·311	−1 055	33	10·8	5 163	2·581	2·582	54	13·9	5·040	4·032	1 008
13	5 8	7 314	7·314	0·000	34	22 5	3·739	1·246	2·493	55	20·4	4·116	3·430	0·686
14	10·6	9 247	9·247	0 000	35	14 6	3·852	0·764	3 088	56	19·7	4·281	2 140	2·141
15	25·1	8 959	6·298	2·661	36	29·4	4·298	1·910	2 388	57	15 5	4·525	1 802	2·723
16	7·3	3·817	1·915	1 902	37	8·4	5·032	3·355	1 677	58	14 8	3 794	2 846	0 948
17	11·2	5 006	3·754	1·252	38	15 0	4·693	2·816	1·877	59	31·9	5 599	4 900	0·699
18	13·7	6·127	6·127	0·000	39	13 3	4·229	4·229	0·000	60	24 5	4·560	3·434	1·126
19	12·4	6·815	4·544	2 271	40	18·2	3·885	3 885	0 000	61	18·0	4·691	5·478	−0·782
20	6 5	8 589	8·589	0·000	41	45 5	7·247	8 153	−0·806	62	21 2	6·626	8·540	−1·914
21	17 1	6·554	5·735	0·819	42	11·5	7·358	6 512	0·846					

Total time $=646^{s}·7$; mean $v=5·816$; mean $v'=5·218$; mean $v-v'=0·598$.

These show plainly both the variation of wind at one anemometer and the difference at the two. In No. 14, $v=9·247$; in No. 10, it is 2·934. These represent V=26·264, and 8·551. If we look to the column $v-v'$, at No. 35 we find +3·088, at No. 62

$-1{\cdot}914$; fourteen are $=0$, and nine are negative. But if we divide them into four consecutive and nearly isochronal groups the discordance is much less.

Time $=\overset{\text{s.}}{161{\cdot}4}$	$v=6{\cdot}219$	$v'=5{\cdot}203$	$v-v'=1{\cdot}016$
$159{\cdot}8$	$5{\cdot}508$	$4{\cdot}375$	$1{\cdot}133$
$160{\cdot}6$	$6{\cdot}241$	$5{\cdot}926$	$0{\cdot}315$
$164{\cdot}9$	$5{\cdot}369$	$5{\cdot}067$	$0{\cdot}302$

The extreme range here is $0{\cdot}831$ instead of $5{\cdot}002$, grouping them in pairs

$T=\overset{\text{s}}{321{\cdot}2}$	$v=5{\cdot}864$	$v'=4{\cdot}791$	$v-v'=1{\cdot}073$
$325{\cdot}5$	$5{\cdot}799$	$5{\cdot}489$	$0{\cdot}310$

There can be little doubt that the total means are nearly correct, and these values of $v-v'$ differ from the mean one by $+0{\cdot}475$ and $-0{\cdot}289$. In general, $v-v'$ will be less than this; and if it be observed by inspecting the chronograph while an observation is proceeding that the ratio of A to A′ varies notably, a longer time should be taken.

(60.) As to the second point it is easily shown that no great errors can arise from assuming that V is truly given by the mean v. The mean V of a series is, taking the time into account,

$$=\frac{\mathrm{SVT}}{\mathrm{ST}}=\frac{x\mathrm{S}v\mathrm{T}}{\mathrm{ST}}+\frac{\mathrm{S}v\mathrm{T}\times\sqrt{z+\frac{\phi}{v^2}}}{\mathrm{ST}}$$

Now the first of these $=x\times$ mean v. In instruments like K where ϕ is small, if we develop the radical in powers of ϕ, the second term becomes

$$\mathrm{S}v\mathrm{T}\times\left(\sqrt{z}+\frac{\vartheta\phi}{2v^2\sqrt{z}}\right)$$

and as the ϕ term may be neglected the mean of radical becomes $\sqrt{z}\times$ mean velocity. When ϕ is large the simplest course is (calling the radical R) to compute $\frac{\mathrm{S}v\mathrm{T}\times\mathrm{R}}{\mathrm{ST}}$, or what comes to the same thing $\frac{\mathrm{S}v\mathrm{CR}}{\mathrm{SC}}$, C being the time-space, and compare this with the R computed with the mean v. Taking at random No. X. of Table XXI. whose $\phi=343{\cdot}28$, we have for the separate groups whose A and A′ are nearly uniform

No	C	v'.
I.	81 8	6·031
II.	95·2	7 411
III.	88·3	7·350
IV.	49 35	4 886
V.	32·5	7·364
VI.	18·68	6 042
	315·83	6·835

$\frac{\text{sum } v\mathrm{CR}}{\text{sum C}}=20{\cdot}633.$

R for mean $v'=20{\cdot}689$.

Requiring the correction $=0{\cdot}056$.

Here the v''s do not range very widely, and I take a more aberrant set observed September 16, 1878, $\phi'=173{\cdot}80$.

No.	C.	v'.
I.	15·38	3·768
II.	107·86	9 537
III.	99·83	4 644
IV.	55 11	3·155
V.	48·47	1 196
VI.	60·66	4 060
VII.	63·16	1·606
VIII.	36·39	1·5925
IX.	54·87	7·393
X.	64 98	5·351
XI.	15·05	0·9625
XII.	16 02	6·331
XIII.	23 00	6 275
XIV.	43 58	4·987
	704·45	4·854

$$\frac{\text{sum } v'\text{RC}}{\text{sum C}}=14·984.$$

R for mean v=14·728.

Correction =+0·156.

Even here the error is not of a nature to interfere with the determination of the constants, though in such work terms like V. and XI. had better be omitted. If it were thought necessary the exact computation is not difficult.

(61.) At first the additional friction was applied by the brake-levers, and was measured by the process described in the note to paragraph (19); but it was soon found to be irregular on account of the rusting of the cast-iron disc on which the rubbers pressed. This could not be prevented in the present location of the instrument.

The rust wore off in the course of an experiment and filled the pores of the cloth on the rubbers. Yet more, it became evident that the constants which in the whirling experiments had given V−W pretty fairly, fail totally here: for instance, with the set last given they give V=14·605, V′=20·066, the difference being far too great to be caused by any error of the friction.

(62.) I intended to remove the uncertainty caused by the rust by substituting for the iron disc one of bell metal of the same diameter; it is, however, some 20 oz. heavier, and the normal friction of E is now 30·4 grains, and its moment =27542. But while it was being prepared it occurred to me that instead of measuring the brake friction first and assuming its permanence during a series of observations it would be better to record and measure it during the entire time of each observation. PRONY's brake afforded a ready means of effecting this, and was thus applied: a ring of iron an inch deep and $\frac{1}{16}$ inch thick is divided into two semicircles held together by screws tapped in the lugs; when these are removed it can be got on the axle, lowered to the disc, and is made to clamp it with any required pressure by replacing and tightening the screws. The ring has an arm which carries an arc of the same depth concentric with the disc and of 8″ radius. It is obvious that when the anemometer is turning, a cord attached to this arc will be pulled by a force =the moment of the ring friction at 8″. This pull is measured by a spring balance which I made with one of the clock springs described in paragraph (26).

(63.) It consists of an iron spindle 6″ long, turning on a small toe below, and above in a brass collar carried by a transverse piece of wood supported on two uprights. It has a projecting piece to which the inner end of the spring is attached by a screw secured by a check nut; the outer end is fixed to one of the uprights. On the top of the spindle is screwed a disc of mahogany 13″ diameter and 0″·5 thick, on the edge of which is turned a groove in which the thread that connects it with the brake arc is wound. On this disc is fitted one of the papers used with my original anemometer, which has its circumference graduated to half degrees, and is covered with circles 0″·05 apart, every tenth one stronger than the rest. By pulling the cord the spindle is turned and the spring tended, the number of its revolutions is shown by a tell-tale fixed to one of the uprights, and the degrees by a pointer.

(64.) It is thus used: tighten the clamp screws so that when the arm is held fast the anemometer shall turn without coming to a stop; pass the cord of the balance through a hole in the remote end of the arc, and tighten it till the increased tension keeps the arm nearly in the same position, then secure it to a pin provided for the purpose. In this state of things it is evident that the tension corresponding to θ, the angle through which the balance has been turned, is the moment of the friction at 8″, from which the moment at the cups is known, to which must be added the normal friction. The brake-ring weighs 14 oz., which would increase the friction a few grains, but this was obviated by hanging an equivalent weight on the relieving apparatus mentioned in paragraph (51). The ring was at first lined with cloth, but as it slightly abraded the bell-metal, I removed it and used the iron surface lubricated with lard.

(65.) The relation between the tension of the spring and θ was thus obtained: the balance being clamped to a table its cord was passed over a pulley; nineteen weights in regular succession from 2 oz. to 36 oz. were hung to it; and to eliminate the effect of friction the disc was turned a few degrees in advance and in rear of the positions of rest, when they were attained, the mean of the θ's was taken as that due to the tension. From ten to thirteen sets were taken for each weight. I had expected that the tension would be very nearly as θ, but with this spring such is not the case; at the beginning $1°=13$ grains, at 4 rev. it $=20$, and the change is not uniform; so I formed from the series an interpolation table with θ argument, from which T is easily computed by a formula analogous to that given by STIRLING for equal intervals.

(66.) In carrying out the work I was met by an unexpected difficulty: friction applied in this way is not constant; and I found that in strong breezes (when the wind is always fitful) the arm oscillated more than 90°, the utmost range which the opening of the iron box (paragraph 2) permitted unless the friction was reduced. These oscillations made it necessary to have a record of the tension, which was provided by clockwork moving a pencil from the centre to the circumference of the graduated paper at the rate of 0″·5 per minute. This traces an irregular sector from which the mean θ is easily obtained. But, besides this, it is necessary to reduce the oscillations below 90°, so that they all may be recorded. This was effected by connecting with

the arc an auxiliar balance, so that its action would begin only at the minor limit of the arm's motion, and increase, so as to prevent it from reaching the major limit. It consists of an iron tube 0″·75 diameter, containing 12″·5 deep of mercury. In this is immersed a rod of iron 0″·3 diameter, reaching to the bottom, and with a cross piece at top resting on the tube; from this cross piece descend two wires carrying weights just sufficient to balance the flotation of the mercury. To the top of the rod is attached a cord, passing over a pulley to the arc. It is easily shown that if the rod be raised an inch the cord will be pulled with a force-weight of a cylinder of mercury 0″·3 diameter and 1″·191 high. (For this also I formed an interpolation table, but in it δ^2 is nearly insensible.)

To use it the spring balance is tended till it keeps the arm near the middle of the opening of the box; the arm is then pressed back to touch the box, the cord is looped on the pin already mentioned and shortened till the cross just rests on the tube, and the θ read which gives the zero of the auxiliar. Deducting this from the mean θ, we have θ', the measure of the auxiliar tension.*

The largest oscillation which I have observed under this arrangement was 54°; the wind was moderate, V being only 16^{m}. This is equivalent to a change of tension $=764$ grains at the cups, nearly 0·4 of the entire tension there. I cannot account for the great irregularities of this friction, but I believe similar facts have been observed on a large scale in applying PRONY's brake to machinery. The extent and frequency of the oscillations do not seem to follow any regular law of V or v, though they evidently are related to them.

(67.) The process described in paragraph (58) gives for each observation an equation containing three unknown quantities, a, x, y, and two unknown variables, V and V′, or $V\pm w$, w being the difference of wind at the two instruments. It is shown by Table XX. that w may be considerable for a few seconds, but when the time is a few minutes it is probably confined within the limits ± 3. Even when (as in the whirling experiments) we know V approximately, and have not V′ to consider, the coefficients are so related that it is impossible to get accurate values of the constants by the usual methods of elimination, and here the difficulty is still greater. I have therefore thought it best to assume probable values for a and y, and determine x so that the mean $V-V'$ may vanish. In the first approximation to this, supposing U the true value of the wind at K, we have $U=V+edx=V\pm w+e'dx$; e being $=\dfrac{V}{\sqrt{z+\frac{\phi}{\iota^2}}}$.

Hence $S(V'-V)\pm Sw=\Delta x\times S(e-e')$.

* A far better mode of retarding the motion of E would be to have on its shaft a sheave connected with an apparatus like that described in paragraph (6), so that the instrument in revolving might draw up the driving weight. The moment of this at the cups would be constant and accurately known, and the observer would only have to continually unwind the cord. Unluckily, this did not occur to me till the series of Table XXI. was nearly completed; and I was unwilling to repeat the measures, for, owing to deficiency of wind, that series had occupied several months.

If the number of observations be sufficient, $S(\pm w)=0$, and we have $\Delta x=\frac{S(V'-V)}{S(e-e')}$. This will give an x nearer the truth (not exact unless $\frac{\Delta x^2}{n}\times\frac{d^2V}{dx^2}$ be insensible), and a second computation will in general be sufficient. When the constants give the mean $V-V'=0$, the V so obtained must be very nearly $=U$, as shall be shown presently.

(68.) First as to α: in the case of $v=0$, we have the measure of it given in paragraph (27). These must be reduced by some hypothesis as to the action of friction. In the first part of my paper I assumed that the momentum of the cups carried them past the point of equilibrium, induced to this by the small value of α given by min. squares (paragraph 39), $=99$. This gives $\alpha=\frac{T-f}{(V-W)^2}$. My preliminary work with the two instruments showed that this was too small, and I recurred to the more natural supposition that the cups stop when the wind's force $=T+f$. This gives $\alpha=15\cdot315$ at Bar. 30° and Therm. 32°.* For 4″ cups it is 3·357, very nearly in the proportion of the areas. I know no means of determining whether this constant varies with v; the individual measures seem to show that it does not change with V. The lateral pressure on the upper bearing of the shaft causes a resistance as V^2, and will diminish α; but the probable value of its coefficient is $\alpha\times 0\cdot 00051$, which may safely be neglected. The change of α, if it exist, cannot have much influence on V; for $\frac{dV}{d\alpha}=\frac{\phi}{2\alpha\sqrt{z+\frac{\phi}{v^2}}}$. Taking I. and X. of Table XXI., where ϕ is a minimum and maximum, we have $dV=d\alpha\times 0\cdot 0573$ and $d\alpha\times 0\cdot 1603$.

(69.) As to y, if there be no resistance as v^2, except what appears in the resultant, the equation of motion is $V^2+v^2-2Vvx-\frac{f}{\alpha}=0$, from which we see that y, the coefficient of v^2, $=1$. This is its major limit; if we diminish y by Δy, the equality of V and V′ may still be preserved by diminishing x. But the value of V is a little decreased: so the $\Delta V=\sqrt{V^2-\Delta y\times vv'}-V$, or, in the case of K, $-\frac{\Delta y\times vv'}{2V}$. Such diminution can only be affected by an expenditure of power in driving air before the cups, or throwing it outwards; and I tried to find a limit by making them revolve in quiescent air. For this purpose I mounted four forms of E on a vertical spindle driven by HUYGHEN's maintaining apparatus, and noted the time and moment at the cups. The resistance was always more than twice the action of direct wind on the convex sides, and I think its excess may be taken as the extreme possible value of y in the

* I tried to measure it by the spring balance, but the oscillations were too extensive to permit any continuous observations. By noting the time, and counting the turns of K while the oscillations of E_2 were clear of the sides of the box, I got two values of α, 15·165 and 19·562; but the possible difference of wind must be remembered.

negative direction. It would give for $y\ \frac{a-r}{a}$; but I think this action must be small in a current of wind moving with nearly three times the velocity of the cups. It is found to increase as the diameter of the cups and the length of the arms diminish; for E it gives $y=+0{\cdot}0546$, but for E_1 (to be soon described) $-3{\cdot}3406$. This supposition would give smaller values of V; for No. II. of Table XXI., where V with $y=1$ is 35·255, the difference is 1·871, and the true value is certainly between these. I will use $y=1$ as certainly known.

(70.) For x, as K and E_1 are similar and equal, it must be the same in both, and the means of obtaining it are explained in paragraph (67). Here I need only show how its first approximation is got: Supposing $w=0$, we have $2Vvx=\frac{\phi'-\phi}{v-v'}+(v+v')$; but as in K ϕ is small, and may here be neglected, we have $V=v(x\pm\sqrt{x^2-1})$, and the sum of the equation becomes

$$2x(\sqrt{x^2-1}+x)\times Sv=S\frac{\phi'-\phi}{v-v'}+S(v+v')\ \text{(VII.)}$$

from which x is easily found. When E is not similar to K the process is simpler: the reading of K gives V, and we have $2xVv'=V^2+yv'^2-\phi'$, whence

$$2x\times Sv=SV+yS\frac{v'^2}{V}-S\frac{\phi'}{V}\ \text{(VIII.)}$$

Both these formulæ are defective from omitting w, but are near enough to begin with.

(71.) The following table gives the results of the comparison of K and E_1, which is equal and similar to K. The second column gives the wind's direction; the third log. air's density; the fourth the time in seconds; the two next A and A′, the number of turns made by K and E_1; the seventh log. of $\frac{f'}{a\times D}$ or ϕ'; the two next the velocities of the centres of K and E; the two next the *computed* velocities of the wind; and the twelfth V′−V. V and V′ were computed by the formulæ $V=v(x+\sqrt{z})+\frac{\phi}{2v\sqrt{z}}$; $V'=v'\left(x+\sqrt{z+\frac{\phi'}{v'^2}}\right)$ when $z=x^2-1$.

TABLE XXI.

No.	Dir.	L. Dens.	Time.	A.	A′.	Log. ϕ'.	v.	v'.	V.	V′.	V′−V.
			s.								
I.	S W.	9·97776	176·1	74	51	1·42156	3·601	2·482	10·291	10 080	−0 261
II.	S.W. b S.	9 97760	391·8	569	511	1 56164	12 444	11·176	35·255	32 471	−2 784
III.	S W.	9 99124	353 9	226	130	1 88336	5 393	3 102	15 327	14 380	−0 947
IV.	E. b. N.	9 99320	350 9	268 3	170	2·07307	6·389	4·047	18 137	18·294	+0 157
V	S W. b S.	9·97684	334 4	322·8	194	2 25147	8 271	4 971	23·453	22 606	−0 847
VI.	S.E. b. E.	9 98104	278 0	215 9	134	1 97602	6 659	4·133	18 874	17 512	−1·302
VII.	"	0 98132	330 9	260 7	161	2·21161	6·749	4 168	19·334	20·397	+1·063
VIII	"	"	210 1	189·5	114	2·27685	7 733	4 652	21·932	22·315	+0·383
IX.	S.E.	9·97757	237·0	306·8	192	2 50054	11 092	6 941	31·428	30 812	−0 616
X.	"	"	176·1	234·3	153	2 53450	10 446	6·835	29·664	31·249	+1 585
XI.	"	"	197·3	213 6	152	2 34007	9 252	6·599	26 228	27·405	+1·177
XII.	"	"	154 4	180	122	2·39493	10 102	6 772	28·631	28·762	+0 131
XIII.	"	"	131 1	157 3	104	2 41366	10 285	6 800	29 149	28 994	−0·155
XIV	N.	9 98888	455 3	369·7	211	2·16403	6 957	3·970	19 741	19 363	−0 378
XV.	S.E. b. S	9·97683	278 6	231·1	160	2·01434	7·108	6 920	20 169	19 441	−0·728
XVI	"	"	259 9	204·4	184	2 10555	6 738	4·417	19 116	19 570	+0·454
XVII	S. b E	9·96418	646·9	639	512	2 13691	8 462	6·780	23·994	25 202	+1 208
XVIII.	S	9 96469	414·3	209	126	2·01367	4·321	2 605	12 813	14 807	+2 494
XIX	S.W.	9 96609	684 9	708 6	489	2 26543	8·867	6·119	25·141	25·293	+0·152
XX.	"	9 96872	404·6	310·3	187	2·10790	6·571	3·960	18·656	18 643	−0 013
XXI.	"	9 95783	713 5	477 5	251 2	1·90599	5 734	3 003	16 290	14 523	−1 767

These were computed with $x=1·5920$ and $z=1·534$; S (V′−V)$=-0·994$, which being divided by S $(e-e)=164·56$, we have $dx=-0·0060$; $x=1·5860$; $z=1·515$; $x+\sqrt{z}=2·826$; log. 0·45111; limit of $\frac{V}{v}=2·826$.

For K, $V=v\times 2·831+\frac{0·355}{v}$.

It is evident from the values of V′−V that the constants do not change with v or v'; but that their differences are casual owing to the difference of wind at the two instruments. They differ when the v's are nearly equal: For instance, I. and VIII. differ by 2·995; VII. and XIV. by 1·441, and IX. and X. by 2·181; and that such differences of wind may exist for some time is shown by Table XX. where during the first 321 seconds V′−V$=-2·888$, and during the following 325s it is $-0·784$.*

The minus values predominate during S.W. winds as might be expected from paragraph (56).

This x and z are larger than those given in paragraph (40), namely, $x=1·2282$, and $z=1·340$, which give for the limit 2·286.

This difference is due partly to my having then used an α only two-thirds of what I believe to be its real value, partly to the uncertainty of the frictions employed and of W, and partly to the defect of the method of minimum squares in such a case.

* As these instruments are generally constructed to register V$=3v'$, their readings should be corrected by subtracting 0 056 of the recorded V.

Reducing the first 21 of Table X. by formula XIII., and with my present values of α and y I get $x=1{\cdot}3744$, $z=0{\cdot}889$, and the limit $=2{\cdot}317$.

The W's used in computing these constants were certainly inaccurate. I measured them in the plane of the centre of the anemometer, but as the disturbance of the air will be as $V \mp v \times \sin\theta$, W must be less in the upper semicircle than in the lower, while it acts with less mechanical advantage in lessening v. It must also be kept in mind that any measure of W is an average one, and that it may have very different values in parts of the air vortex

(72.) In E_2, the cross remaining the same the 9″ cups were set at 12″ from the axis; it is my No. III. In it the constant for v' is half that for K's v, and the normal friction is double $=60{\cdot}8$. With the approximate $x=1\ 7481$ and $z=2\ 056$, the results are given in the following Table, in which the densities are omitted as involved in ϕ'.

TABLE XXII.

No	Dir	Time	A	A′	Log ϕ'	v	v'	V	V′	V − V′
		s								
I	N E	474	108	165	0 67241	0 976	1 491	3 151	5 590	−2 439
II	N E	605 6	318	540 5	0 62808	4 434	3 828	12 637	12 539	+0 098
III	S E	541 7	281 5	505 5	0 64167	4 458	3 998	12 693	13 093	−0 400
IV	S E b S	550 9	570	1011 0	0 63633	8 864	7 861	25 135	25 206	−0 071
V	S W b S	503 6	208 0	342 0	0 63394	3 453	2 909	9 795	9 853	−0 058
VI	S E	446 2	261 6	458	0 63185	5 0235	4 350	14 298	14 179	+0 119
VII	S	601 3	611	1134	0 63300	8 705	8 078	24 686	25 891	−1 205
VIII	S	547 1	440 3	752	0 63300	6 895	5 888	19 562	18 971	+0 591
IX	S W	607 8	453 5	782	0 63397	6 348	5 551	18 020	17 779	+0 241
X	W	552 1	380	647	0 63127	5 896	5 019	16 754	16 083	+0 671
XI	S W	599 4	133	240	0 63644	1 914	1 727	5 617	6 254	−0 637
XII	S W	475 8	340	576 5	0 63295	6 121	5 190	17 389	16 797	+0 592
XIII	S W	480 8	330	582	0 63966	5 345	5 185	15 201	16 624	−1 423
XIV	W b N	572 1	192	331 5	0 63283	3 212	2 773	9 212	8 938	+0 274
XV	N W	663 5	314	535 5	0 62821	4 045	3 4495	11 584	11 381	+0 203
XVI	N W	515 2	290	497	0 62623	4 3765	3 674	12 474	12 068	+0 406
XVII	N W	600	248 2	412 5	0 62710	3 544	2 945	10 139	9 845	+0 294
XVIII	N W	660	178	318 8	0 62180	2 311	2 037	6 704	7 127	−0 423
XIX	S W	946 6	488 7	752	0 63008	4 4205	3 401	12 598	11 243	+1 355
XX.	S	720	354	599	0 63364	4 212	3 564	12 014	11 453	+0 561

$S(V-V')=-1\ 251$ which divided by $Se'=145\ 185$, we have $dx=-0{\cdot}0086$ $x=1\ 7395$, $z=2{\cdot}026$, and the limit $=3\ 163$ These are larger than those of E_1. The results obtained in paragraphs (38) and (39) would make it less, but in the present work it is the rule that diminishing the length of the arms increases x

(73) In E_3 the 9″ inch cups were fixed 8″ distance from the axis. too near for good work, but I wished to see the effect on x. With its approximate values $x=2\ 1359$ and $z=3{\cdot}562$, I computed Table XXIII.

TABLE XXIII

No	Dir	Time	A	A	Log ϕ	v	v	V	V	V−V
		s								
I	S	660	348 5	677 5	0 81086	4 524	2 931	12 891	12 349	+0 542
II	W b N	475 4	181	237	0 80286	2 859	1 423	6 836	6 721	+0 115
III	W b N	600	238	433	0 80390	3 399	2 061	9 731	9 037	+0 694
IV	W	600	192	360	0 80174	2 742	1 714	7 900	7 761	+0 139
V	S F	540	458	940	0 81188	7 267	4 972	20 623	20 424	+0 199
VI	S b E	646 9	494 5	1011 3	0 81938	6 631	4 972	19 088	20 343	−1 255
VII	S W	334 8	227 3	471	0 80834	5 883	4 063	16 718	16 757	−0 039
VIII	S b W	600	300 5	599	0 80480	4 291	2 851	12 233	12 032	+0 201
IX	S S W	600	260	465 5	0 82025	3 718	2 216	10 610	9 629	+0 987
X	S S W	720	540	1047	0 81604	6 426	4 153	18 251	17 115	+1 136
XI	S W	600	286	593 5	0 81637	4 084	2 825	11 682	11 927	−0 245
XII	S W	600	285	609	0 81150	4 070	2 898	11 596	12 223	−0 627
XIII	S W	420	316	638	0 8149[illegible]	6 446	4 336	18 307	17 623	+0 684
XIV	S	600	466 5	910	0 81140	6 662	4 331	18 915	17 817	+1 098
XV	W	720	291	574	0 80694	3 462	2 277	9 910	9 839	+0 071
XVI	N W	600	235 5	445	0 80694	3 362	2 118	9 631	9 255	+0 376
XVII	S W b W	511	463 3	959 5	0 79931	7 767	5 363	2 033	21 883	+0 150
XVIII	S W b W	509 1	395	888 7	0 79712	6 960	4 9 7	20 029	20 271	−0 242
XIX	S W	600	350 3	727	0 80133	5 002	3 460	14 235	14 389	−0 154
XX	S b W	491 9	312 5	751	0 80780	5 448	4 361	15 477	1 937	−2 460

The sum of V−V $=+1\ 370$, $Sc=119\ 53$ Hence $\Delta x=+0\ 0114$, $x=2\ 1473$, $z=3\ 611$ and the limit $=4\ 047$ The residue is a little too large, but I did not think it necessary to pursue the approximation farther The great increase of x is remarkable, and I think shows that when the cups are so near the axis of rotation they disturb the regular action of the wind Even with the 12″ arms this effect is sensible

(74) E_t I now fixed the 4 cups on the cross at 10 $\frac{2}{3}$ from the axis This arrangement is *similar* to the 9″ cups at 24, and I thought that the same x might serve for both, but it was far otherwise

The measures in paragraph (27) give for 4″ cups $\alpha=3\ 357$ at the normal pressure and temperature For the first ten observations $f=35\ 19$ but as these cups are 35 5 oz lighter than the 9 ones, I thought there was too little pressure on the toe, and changed the relieving weight from the 11 5 lbs to 9 lbs This made the friction $=68\ 89$ I computed with $x=2\ 57$ and $z=5\ 405$ Table XXIV

TABLE XXIV

No	Dir	Time	A	A	Log φ	ı	v	V	V	V−V′
		s								
I	S W	600	291 5	343	1 04878	4 156	2 177	11 853	11 719	+0 134
II	S W	420	202 5	211	1 04675	4 131	1 946	11 785	10 516	+1 219
III	S W	225 03*	188 6	147	1 05066	5 526	2 488	15 710	13 169	+2 541
IV	S W	588 7	433	554 5	1 05884	6 302	3 587	17 919	19 696	−1 777
V	W b N	540	156	139 5	1 04918	2 475	0 990	7 159	6 676	+0 483
VI	S W	600	313 5	414 5	1 04366	4 477	2 631	12 750	13 362	−0 612
VII	S W	600	263	306 3	1 04293	3 756	1 941	10 730	10 674	+0 056
VIII	S b W	780	391 7	449	1 04205	4 303	2 192	12 266	11 796	+0 470
IX	S b W	600	371 5	449	1 04491	5 305	2 850	15 096	15 314	−0 218
X	S S W	660	402	499	1 04389	5 219	2 879	14 844	14 969	−0 125
XI	S b E	840	430 3	471	1 33626	4 389	2 185	12 509	12 575	−0 066
XII	W S W	532 3	261 1	290 4	1 33929	4 145	2 078	11 824	12 112	−0 288
XIII	S W b S	600	420 3	535 5	1 33648	6 002	2 085	17 052	18 201	−1 149
XIV	S W b W	600	419 5	503	1 33768	5 949	3 193	17 019	17 117	−0 098
XV	S W	608 1	362 3	408 5	1 33538	5 105	2 558	14 524	14 211	+0 318
XVI	N	540	320 5	407	1 32198	5 085	2 870	14 470	15 480	−1 011
XVII	N	600	363 5	430	1 32039	5 191	2 729	14 763	14 933	−0 170
XVIII	N	600	382 5	493	1 32025	5 462	3 129	15 527	16 741	−1 214
XIX	W	666	493 5	512 5	1 33003	6 407	? 957	18 193	15 991	+2 202
XX	W b N	600	461 6	550 5	1 33073	6 592	3 494	18 714	18 456	+0 258

The sum of V−V $=+0\ 948$, $Sc=83\ 121$, so $\Delta x=+0\ 0114$, $x=2\ 5814$, $z=5\ 664$, and the limit $=4\ 961$ This great excess of x above that of K is very remarkable, and shows not only that anemometers must be equal as well as similar to have the same constants, but also that x depends on the diameter of the cups as well as the length of the arms, for here it is greater than in E_3, though the arms are longer

(75) E_3 The 4 cups were fixed as far out on the cross as possible, the distance from the axis being 26 75, 2 75 greater than that of K, and I expected its x would be less At first it was mounted on the axle of E, but it moved so much slower than K that I thought its friction $=27\ 42$ was too much for the small cups I therefore mounted it on the spindle already mentioned with friction $=4\ 72$, and with *its* results (VI, XXIII) computed Table XXV

TABLE XXV

No	Dir	Time	A	A′	Log φ	v	v	V	V	V−V
		s								
I	S	600	547	378	0 94639	7 812	6 016	22 163	21 194	+0 969
II	S W b S	600	444 5	292 3	0 94689	6 347	4 652	18 026	16 617	+1 409
III	W b N	600	302	172 5	0 92562	4 813	2 825	12 293	10 599	+1 694
IV	N W	300	261 8	188	0 92552	7 477	5 825	21 207	20 527	+0 680
V	S W	600	547	387	0 94124	7 811	6 160	22 159	21 674	+0 485
VI	S W	660	646	484	9 97483	8 386	7 008	23 783	24 291	−0 508
VII	S W	660	728 5	561	0 16863	9 3925	8 117	26 627	28 134	−1 507
VIII	S W	600	679	518	0 16820	9 696	8 300	27 485	28 300	−0 815
IX	S W	600	309 8	209 5	0 17272	4 424	3 334	12 609	11 616	+0 993
X	W	1200	922	660 5	0 16968	6 583	5 256	18 689	18 269	+0 420
XI	W	480	250	162 25	0 16849	4 452	3 288	12 686	11 512	+1 174
XII	S W	600	521	384	0 16846	7 440	6 112	21 110	21 219	−0 109
XIII	S W	600	458	320 5	0 16785	6 540	5 101	18 570	17 736	+0 834
XIV	S W b S	600	473	320	0 16788	6 754	5 093	19 174	17 710	+1 464
XV	S	600	432	314 5	0 16727	6 056	5 007	17 205	17 412	−0 207
XVI	S	600	555	437 5	0 16864	7 925	6 968	22 481	24 450	−1 969
XVII	S E	600	388	302	0 15962	5 542	4 817	15 749	16 720	−0 971
XVIII	S E	600	357 3	268	0 16041	5 103	4 266	14 514	14 870	−0 356
XIX		1200	908	725	0 16274	6 483	5 770	18 408	20 087	−1 629
XX	S W	C60	659	495 5	0 17775	8 555	7 169	21 261	22 902	−1 641
XXI	W b S	600	417	291 5	0 17795	5 956	4 639	16 920	16 152	+0 768
XXII	S W	600	459 5	322	0 17390	6 561	5 125	18 682	17 883	+0 799
XXIII	S W	600	362 5	264	0 17404	5 176	4 202	14 726	14 645	+0 081

* Time short because it was at the end of the chronograph sheet

Omitting the first five $S(V-V')=-3{\cdot}179$, $Se'=269{\cdot}80$. Hence $\Delta x=-0{\cdot}0117$ and $x=1{\cdot}8624$ and $z=2{\cdot}468$ and the limit $=3{\cdot}436$. This result surprised me, for the friction was so small that no irregularity of it could have any sensible influence, nor does it seem probable that the pressures on the surfaces of the two sets of cups are in any other ratio than that of the surfaces. The x is actually larger than that of E_2. The five first V''s were computed with the final x. They give V' rather too small, but in three of them the wind was S.W.

(76.) E_6. I now placed my old anemometer, cups 12'', arms 23''·17, on the axle of E. The α of these cups (if as their area)$=27{\cdot}227$* and their $f=29{\cdot}0$. With the second approximation, $x=1{\cdot}5897$, $z=1{\cdot}527$, I recomputed the V and V' of Table XXVI.

Table XXVI.

No.	Dir.	Time.	A.	A'.	Log. φ'.	v	v'.	V.	V'.	V – V'.
		s								
I.	N.W.	600	313	339	0 05547	4 470	4 674	12 737	13 415	−0 678
II.	N.	600	332·5	386 5	0 04264	4·747	5 328	13 519	15 133	−1·614
III.	N.	1200	695	784	0·05223	4 962	5 404	14 122	15 351	−1 229
IV.	N b E.	600	240	286	0 04524	3·427	3 943	9 809	11 247	−1 438
V.	N.E.	600	382	378	0 04701	5 455	5 211	15 509	14·806	+0·703
VI.	W b N.	600	292	300	0 05411	4 170	4·136	11 896	11·795	+0 101
VII	S W.	600	398	368	0 05435	5·683	5 074	16·379	14 404	+1·975
VIII.	S E.	600	751	783	0 05278	10·725	10 796	30 392	30 464	−0·072
IX.	S	600	726	751 7	0·05918	8 640	8·636	24 499	24 484	+0 015
X.	S.	660	620	615·8	0 05918	8 048	7·778	22 830	22 014	+0 816
XI.	S	660	659	679 6	0 05918	8 555	8 517	24·261	24 120	+0 141
XII.	S.	660	502	516·7	0 05918	6·516	6·476	18 506	18 365	+0 141
XIII.	N.E	600	286	282	0·04772	4 277	3 887	11 652	11 591	+0 061
XIV.	E.	600	340	329	0 04556	4 855	4 535	15 497	12 900	+2 597
XV	E	600	358	348 5	0 04833	5 112	4 935	14 454	14 037	+0 417
XVI.	E.	1200	793	860	0·04435	5 662	5 928	16 093	16 890	−0 797
XVII.	N.E. b. E	600	279 7	294 5	0 04666	3·994	4 060	11·390	11·578	−0·179
XVIII.	N E. b. E.	600	278·5	277 8	0 04797	3 977	3 830	11 351	10 934	+0·417

It will be observed here, as in Table XX., that v' is sometimes greater, sometimes less than v; the near equality of the constants of the 12'' cups to those of K makes the irregularities of the wind manifest.† The $S(V-V')$ giving III. and XVI. (double weight) $=-0{\cdot}649$, $Se'=282{\cdot}65$, therefore $dx=-0{\cdot}0023$, $x=1\ 5874$, $z=1{\cdot}520$, and the limit 2·8202. This x is a little less than that of K; which shows that the influence of the diameter of the cups is felt even here, overpowering the effect of the shorter ones.

(77.) I shall conclude with a few remarks on the preceding results.

The process by which the x of K is determined seems liable to but two objections:

* In my original paper "On the Cup Anemometer" (Trans. R I A., vol. xxii , p 170) I have mentioned some trials to measure α. As the V's given there were doubtful, I have recomputed them with these constants and the friction of that instrument $=48{\cdot}61$. The six give for α (at normal pressure and temperature) 27·898, agreeing fairly with that given in the text.

† I may mention here, as further proof of the unsteadiness of the wind, that on one occasion I reversed two cups of this anemometer so that all the convexes were opposed to the wind; I expected they would remain at rest, but they were in continual oscillation through many degrees, so that in the limited area $5'\times 1'$ there must have been differences of V able to overcome a friction of 53 grains.

the assumption that $y=1$, and the possibility that the wind errors are not eliminated. As to the first, I have shown in paragraph (69) that it cannot be far astray; even were the extreme diminution of it which is mentioned there to occur, the error for $V=100$ would only be 6·1; but such is not likely to be the case, and $y=1$ may be accepted as a major limit and one not far from the truth. As to the second it is certain that in a sufficient number of observations the $+$ and $-$ errors must balance each other; but it may be a question whether the XXI. of Table XXI. were enough. Still, it is evident, from inspection of the column $V-V'$ that there cannot be any large outstanding residue. I have pointed out the defects of the situation. Could I have had my wish I would have established the instruments on spars 20′ high, erected on a level ground away from any influence of houses or trees, and used a better mode of applying friction to E_1. I also regret that no strong gale occurred during these experiments to verify the formula for a very large v. Under favourable circumstances, I think this mode of determining x the best that is known. I have stated reasons for distrusting the results obtained by the whirling machine, and as yet no unexceptional mode of carrying an anemometer through the air in a right line has been devised. Even could we get a locomotive which could travel without disturbing the air through which it passes (and perhaps the new electric motors might effect this), and a line of rails certainly screened from wind, there would still remain the doubt whether the pressure is the same when a body is moved through a quiescent fluid or a current impinges on the body.

(78.) The results obtained with other anemometers show that x is a function both of R, the length of the arm, and r, the radius of the cups. I subjoin its values.

$R=23·17$	$r=6$	$x=1·5880$
24	4·5	1·5919
12	4·5	1·7463
8	4·5	2·1488
26·75	2	1·8587
10·67	2	2·5798

If we take Nos. 2, 3, and 4 in which the cups are equal, the dependence of x on R is manifest. In No. 3 it is $\frac{1}{10}$ larger than No. 2, and in No. 4 $\frac{1}{3}$. This is partly due to the air's escape before the convexes being less easy as the circle described by them is less. Such a fact is strikingly shown by the whirling experiments (paragraph 69) which I made in search of a minor limit for y, in which I found the resistance $=30·61$, and 79 for the three respectively. This was in quiescent air; but a similar though much smaller effect must occur in the actual working of an anemometer. Its influence can be obtained only by experiments, such as the present.*

* I thought to test this by removing two opposite cups in E_3. As in this case there is only one cup in each semicircle at a time, the probability of their mutual disturbance was small. A set of ten gave the

(79.) It is more difficult to account for the similar dependence of x on the size of the cups; *à priori*, there seems no reason why small cups should be more resisted than large ones, but such is evidently the case. Unluckily I did not place the 12″ and 4″ cups at the same distances as the 9″, so that the effect of the cups on x is mixed with that of R. I tried to eliminate the latter by interpolating for the values that the 9″x would have at the R's of Nos. 1, 5, and 6, but this could not be done very exactly from the three values. However, this gives x for the 12″ 0·005 less than for the 9″; for the 4″, in No. 5, 0·2894 greater, and in No. 6, 0·7517. The only way in which I can conceive the possibility of such an occurrence is the existence of powers of r and R in the factors, which express the mean effects of wind on the concave and convex surfaces of the cups. In equation III. I suppose the mean v to be that of the centre of the cups, and that the mean impulse and resistance act at these points. But this is not necessarily the case. The effect of the resultant on an element of the cup is (1) as the square of that resultant; (2) as the perpendicular pressure on the element; (3) as the resultant of that pressure perpendicular to the plane of the cup's mouth; (4) as the distance from the axis at which the projection of that resultant meets the arm; and (5) as the magnitude of the element. Of these five factors the first contains v and v^2. Now v as the element $=v\times\frac{(4)}{\mathrm{R}}$ which contains R and r; r^2 also enters the fifth, so that the differential may contain R^3 and r^5. As to the second we are ignorant of its formula, and it is pretty certain that it will depend on powers of the sine and cosine of incidence and (at least for the concave) on the curvature. If we knew its exact form we could integrate the differential which they form and get the impulse and resistance for a given θ, and multiplying this by $d\theta$, and again integrating from 0 to π we should find their mean values. Of the terms in this last integration those which have $\sin^2\theta$ as a factor disappear; πr^2 (the surface) will be a factor of the others, among which *may* be the three first powers of $\frac{r}{\mathrm{R}}$; and these may produce the change of x.

(80.) In paragraph (41) I inferred from the work with the whirling machine that with 9″ cups the x is the same for 24″ and 12″ arms; but what precedes shows that this is not the fact, and that each type of anemometer has a special x. I would therefore suggest to meteorologists and opticians the propriety of confining themselves to two types: one for fixed instruments, the other for portable ones. For the first the Kew type should, I think, be adopted; if the determination of its constants, given in paragraph (70), be not thought sufficiently exact, there would be little difficulty in making more observations like those described there, and under more

$x=2·0709$ less than in No. 4, but so large as to make it evident that there must be some other cause of the increased value of x.

favourable circumstances; but I think they would make very little change in my number. For the portable instrument, the only one of which I have experience is CASELLA's 3″ cups and 6″ arms, and I found it very convenient: its x should be determined as above. Some such arrangement seems necessary to ensure a uniformity of velocity measures.

XXVI. *On the Dynamo-electric Current, and on Certain Means to Improve its Steadiness.*

By C. WILLIAM SIEMENS, *D.C.L., F.R.S.*

Received March 1,—Read March 4, 1880.

[PLATES 40–52.]

ON the 14th February, 1867, I communicated a short paper to the Royal Society, describing the accumulative or dynamo-electrical principle of action, the conception of which I attributed to my brother Dr. WERNER SIEMENS. When the paper was read, another paper followed by Sir CHARLES WHEATSTONE (sent in on the 24th February) also describing this principle of action, thus showing that the same line of thought had occupied that eminent philosopher.

In illustration of my paper I exhibited a machine of my design, embodying the accumulative principle of action, which furnished abundant evidence of the powerful nature of the current that could be thus produced. It consisted of two horseshoe electro-magnets, between the poles of which a SIEMENS armature could be made to rotate, the machine being furnished with a handle or pulley for that purpose. A commutator was provided, by which the alternating currents set up in the rotating coil (after a first impulse had been given) were directed through the coils of the stationary electro-magnets in a continuous manner, and proceeded thence outward to ignite a platinum wire of some 12″ in length, or to perform other work.

This machine, although the first of its kind, has done good service ever since its construction, having been found very efficacious in exciting powerful permanent magnets at the telegraph works of SIEMENS Brothers at Woolwich.

Since 1867 the accumulative principle has been employed in the machines of different makers, and one form of dynamo-electric machine, that of M. GRAMME, differs very materially from the machine above referred to, and has met very deservedly with extensive recognition. M. GRAMME embodied in his machine the principle of Professor PACINOTTI's magnetic ring, which enabled him to produce powerful electric currents without much of the loss of energy caused in previous machines through the heating of the rotating armature.

Another modification of the dynamo-electrical machine is one devised by Mr. Von HEFTNER ALTENECK, an engineer and physicist employed under my brother WERNER SIEMENS, at Berlin. This machine differs from that first submitted by myself in several important particulars. Instead of the WERNER SIEMENS armature, Von HEFTNER ALTENECK adopted a rotating coil of iron wire wound with insulated copper wire in more than one direction, the several coils of wire being connected seriatim

with the commutator, and through it, with the wire surrounding the soft iron bars, and with the electric lamp or other resistance on the outer circuit.

The advantage claimed for this mode of construction is that all the wire forming the rotating coil or helix is brought into the magnetic field, excepting only those portions crossing from side to side of the coil; and in order to reduce this unproductive resistance to a minimum, the rotating coil or helix has been made comparatively long, and the number of electro-magnets has been increased generally to six or more.

The principal advantage of the dynamo-electrical machine over all other current generators consists in its power of producing currents of great magnitude, and of an intensity up to 100 volts, with a small primary resistance, and therefore with a comparatively small expenditure of mechanical energy. It labours, on the other hand, under the disadvantage that the power of the current depends, at a given velocity, upon the magnetic force developed in the electro-magnets. This force depends upon the amount of current passing through the coils of the magnets, which in its turn is dependent in an inverse ratio upon the resistance in the outer circuit. If from some accidental cause the external resistance is increased, the electro-motive force of the machine, instead of rising to overcome the obstruction, diminishes, and thus aggravates the resulting disturbance. If, on the other hand, the resistance of the outer circuit diminishes, as in the case when the carbons of an electric regulator touch one another, the electro-magnets are immediately excited to a maximum, and the electro-motive force of the machine is increased. The power absorbed and its equivalent, the heat generated in the circuit, is equal to the square of the electro-motive force divided by the resistance; hence the work demanded from the engine will be greatly increased, the machine may be dangerously overheated, and powerful sparks may injure the commutator. It is chiefly owing to this instability of the dynamo-electric current that its application to electric illumination has been retarded, and that magneto-electric machines and machines producing alternating currents have been again used, although they are inferior to the dynamo machine in the current energy produced for a given expenditure of mechanical energy.

The properties of dynamo-electric machines have been examined by several observers. Messrs. HOUSTON and THOMSON (Franklin Institute) compared the efficiency of the GRAMME, BRUSH, and WALLACE FARMER machines. Dr. HOPKINSON (Institution of Mechanical Engineers, 25th April, 1879) examined a medium-sized SIEMENS machine, determined its efficiency, and expressed the electro-motive force as a function of the current. Herrn MAYER and ANERBACH (WIEDEMANN'S 'Annalen,' November, 1879) experimented on a GRAMME machine, and obtained a curve very similar to Diagram I. M. MASCART has experimented on the GRAMME machine, and Mr. SCHWENDLER on both GRAMME and SIEMENS machines.

The radical defect of the dynamo machine of ordinary construction, may be inferred from the results of these experiments. The remedy has, however, been in our hands from the time of the first announcement of the principle of these machines before the Royal Society, when Sir CHARLES WHEATSTONE pointed out that "a very remarkable

increase of all the effects, accompanied by a diminution in the resistance of the machine, is observed when a cross wire is placed so as to divert a great portion of the current from the electro-magnet."

Some of the constructors of dynamo machines, namely: Mr. LADD in this country, and Mr. BRUSH in the United States of America, have taken advantage of this suggestion, the latter with the avowed object in view of obviating spontaneous changes of polarity in effecting electro-precipitation of metals, and without perhaps having realised all of the advantages of which this mode of action is capable; others have refrained from doing so on account of difficulties resulting, as I shall endeavour to show, from an insufficient examination into some important physical conditions that require attention in order to realise economical results.

An ordinary medium-sized SIEMENS-ALTENECK dynamo-electrical machine has wound on its rotating helix insulated copper wire of 2·5 m.m. diameter in 24 sections, representing a resistance of ·4014 S. U.* The four electro-magnet coils connected seriatim are composed of copper wire of 5·5 m.m. diameter, presenting a total resistance of 0·3065 S. U.

If (as has frequently been done) the wires of this machine were to be connected as suggested in Sir CHARLES WHEATSTONE'S original paper, thus making the outer circuit not continuous with but parallel to the coil circuit, and if the outer circuit had a resistance of one unit, it would follow that the total resistance to the current set up by the rotation of the armature would be reduced from $\cdot 4+\cdot 3+1\cdot=1\cdot 7$ to $\cdot 4+\frac{\cdot 3\times 1}{1+\cdot 3}=0\cdot 61$ unit, causing a great increase of current, the major portion (in the proportion of 10 to 4) would flow through the electro-magnets, thus causing a great increase of heating effect. The resistance of the field magnet must therefore be greatly increased, but if it were attempted to increase that resistance simply by reducing the diameter of the wire, and increasing the number of convolutions until the same thickness of coil was obtained, the magnetic excitement and with it the electro-motive force of the current produced at a given velocity of rotation would suffer a material decrease. The current flowing through the helix coil would moreover have to divide itself, and in order to reach the same limit in the outer circuit its intensity in the helix coil would have to be increased, causing it to heat more readily than before. It was necessary, therefore, to raise the effect of the magnet current to the same level as before with as small a proportion of the helix current as possible, in order to leave a maximum proportion of the current for the outer circuit. In order to effect this, the magnet bars had to be increased in length, and placed further apart so as to provide room for

* The resistance coils used in these experiments were graduated according to the mercury system introduced by Dr. WERNER SIEMENS, and adopted by the Telegraphic Convention at Vienna in 1868. The B. A. unit was determined in 1874 by KOHLRAUSCH to be 1·0493 S. U., or combined with LORENZ'S value of the S. U. afterwards adopted, $0\cdot 9797\times 10^9$ C. G. S. units—as much as 2 per cent. below its ascribed theoretical value. Later determinations by H. F. WEBER (Phil. Mag., March, 1878) makes the S. U. to be equal to $0\cdot 955\times 10^9$ C. G. S. units, and thus the ohm to be 0·2 per cent. higher than its ascribed value; if this latter value is used, the numerical results must be correspondingly altered.

coils of greatly increased weight and dimensions; at the same time the helix wire had to be increased in diameter to give room for the aggregate current, but in reality I found it advantageous to increase the diameter of the same in a much greater proportion.

These general conditions having been determined by preliminary experiment, Mr. LAUCKERT, electrician engaged at my works, undertook a series of comparative experiments which are given in the appendix attached to this paper, and the results are given numerically and exhibited in curves. On examining the curves it will be remarked:

1. That the electro-motive force instead of diminishing with increased resistance, increases at first rapidly, then more slowly towards an asymptote.

2. That the current in the outer circuit is actually greater for a unit and a-half resistance than for one unit.

3. With an external resistance of one unit, which is about equivalent to an electric arc when 30 or 40 webers are passing through it, 2·44 horse-power is expended, of which 1·29 horse-power is usefully employed: an efficiency of 53 per cent. as compared with 45 per cent. in the case of the ordinary dynamo machine.

4. That the maximum energy which can be demanded from the engine is 2·6 horse-power, so that but a small margin of power is needed to suffice for the greatest possible requirement.

5. That the maximum energy which can be injuriously transferred into heat in the machine itself is 1·3 horse-power, so that there is no fear here of destroying the insulation of the helix by excessive heating.

6. That the maximum current is approximately that which would be habitually used, and which the commutator and collecting brushes are quite capable of transmitting.

Hence I conclude that the new machine will give a steadier light than the old one, with greater average economy of power, that it will be less liable to derangement, and may be driven without variation of speed by a smaller engine; also that the new machine is free from the objection of having its currents reversed when used for the purpose of electro deposition.

The same peculiarity also enables me to effect an important simplification of the regulator to work electric lamps, to dispense with all wheel and clock-work in the arrangement, as shown in Plate 40. The two carbons, being pushed onward by gravity or spring power, are checked laterally by a pointed metallic abutment, situated at such a distance from the arc itself that the heat is only just sufficient to cause the gradual wasting away of the carbon in contact with atmospheric air. The carbon holders are connected with the iron core of a solenoid coil, of a resistance equal to about fifty times that of the arc, the ends of which coil are connected with the two electrodes respectively. The weight of the core, which has to be maintained in suspension by the attractive force produced by the current, determines the distance between the electrodes, and hence the electric resistance of the arc. The result is that the length of the arc is regulated automatically so as to maintain a uniform resistance, signifying a uniform development of light.

APPENDIX.

The measurements of the electric currents were made with an electro-dynamometer, the movable part of which consisted of a single turn of 4 m.m. wire, and the stationary coil of nine turns of the same.

To be able to reduce the electrical measurements into absolute power developed, it was in the first place necessary to determine the constant of the instrument in use. This was done in the following manner:—Five copper plates of about 11″ × 8″ were connected as shown in the sketch.

These were carefully weighed and immersed in a solution of sulphate of copper. The machine was previously started, the time of immersion carefully noted, and the readings of the current taken every half minute. The plates were so arranged that the current entering at *a* and leaving at *z* deposited the copper on both sides of the plate at *z*. After a certain time the plates were taken out, quickly rinsed in water, and dried in sawdust. The plates were then carefully weighed again and the deposit calculated per degree reading on the instrument per second of time. Six independent measurements were taken with currents varying from 20 to 40 webers, and gave a mean of ·000779 gramme of copper per second per degree reading. The differences of these measurements from the mean varied from 0·21 per cent. to 6·6 per cent., the mean of the differences being 1·98 per cent.

According to F. KOHLRAUSCH (POGG. Ann., Bd. cxlix., 1873) the quantity of silver deposited by the C. G. S. unit of electricity is 0·011363 gramme, and since the quantities vary as the equivalents of the metals deposited, we have

$$\frac{\cdot 011363 \times 63 \cdot 5}{216} = 0 \cdot 003340 \text{ gramme of copper.}$$

One weber being $\frac{1}{10}$ C. G. S. unit, we have to divide by 10 the quantity of copper deposited by a current of one weber in one second, that is ·000334 gramme, and dividing ·000779 by ·000334 we get 2·23323 webers for a degree reading of our instrument.

To be able to compare the machines having the new winding (*i.e.*, the wire on the electro-magnets connected parallel with the outer circuit) with the ordinary machines, it was necessary to experiment on the relation existing between the power expended and the current produced with different resistances in circuit and different speeds.

A medium dynamo machine with 24 part commutator was used, the helix being wound with 336 convolutions of 2·5 m.m. wire, having a resistance of ·4014 S. U. when measured in the machine. The electro-magnets were wound with four layers of

5·5 m.m. wire, each having 32 convolutions, and therefore the four bobbins a total of 512 convolutions with a resistance of ·3065 S. U.

The accompanying Tables Nos. 5, 6, 7, 8, and 9 give the details of the experiments made, which are shown graphically in the diagrams similarly numbered. The current in webers was simply calculated by multiplying the square root of the reading on the electro-dynamometer with the constant of the instrument, *i.e.*, 2·3323.

To be able to calculate the electro-motive force from the current in webers and resistance in SIEMENS' units, it was necessary to convert the S. U. into C. G. S. units by multiplying the same by $\cdot 9337 \times 10^9$. (This figure is given by LORENZ, POGG. Ann., Bd. cxlix., 1873.) By again multiplying this resistance into the current we get, according to OHM's law, the electro-motive force in C. G. S. units, and by dividing by 10^8 we get the E. M. F. in volts.

I have further calculated the total amount of work developed in the following manner:—

Work done $= E \times C \times t$, or, which is the same, $C^2 \times R \times t$, where E is E. M. F; C, current; R, resistance; t, time.

From these calculations t is eliminated as it occurs in all the equations.

$$1 \text{ volt} = 10^8 \text{ C. G. S. units.}$$
$$1 \text{ weber} = \tfrac{1}{10} \text{ C. G. S. unit of current.}$$
$$1 \text{ HP} = 7{\cdot}46 \times 10^9 \text{ C. G. S. units.}$$

Therefore

$$\frac{1 \text{ volt} \times 1 \text{ weber}}{1 \text{ HP}} = \frac{10^8 \times 10^{-1}}{7{\cdot}46 \times 10^9} = \frac{1}{746}$$

and if we multiply the E. M. F. in volts by the current in webers, and divide by 746, we have the actual work developed in horse-power.

To find the actual work done in the outside resistance we use the formula $C^2 \times R$, of course having to reduce the resistance R into absolute C. G. S. units by multiplying by $\cdot 9337 \times 10^9$.

The machine with the new winding had a helix with 24 part commutator wound with 312 convolutions of 2·8 m.m. wire.

The electro-magnets being lengthened by 2″ to take bobbins 10½″, instead of 8½″ as on the ordinary machines, I had three sets of bobbins made, and had the same wound with different sizes of wire, viz.: 2·5 m.m., 2·8 m.m., and 3 m.m., having a respective resistance of 11·26, 7·563, and 4·46 S. U.

The accompanying Tables Nos. 1, 2, 3, and 4 show the experiments made with this machine with electro-magnets of 11·26 S. U. resistance; Nos. 10, 11, 12, and 13 with electro-magnets of 7·563 S. U.; and Nos. 14 and 15 with electro-magnets of 4·46 S. U. The helix in all cases having been wound with 2·8 m.m. wire with a resistance of ·234 S. U. when measured in the machine.

The Tables marked 5, 6, 7, 8, and 9 refer to the dynamo machine wound in the ordinary way.

The Tables marked 16, 17, 18, 19, 20, 21, 22, and 23 show the results obtained with a machine having a helix wound with 288 convolutions of 3 m.m. wire and a resistance of ·173 S. units. The electro-magnets, as before, had a resistance of 11·26, 7·563, and 4·46 S. U.

No. 1.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.

Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance (connected parallel to outer circuit).

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E. M. F. in helix in volts	Horse power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit	Helix.	Electro-magnet.	Outer circuit	Helix	Electro-magnet	Outer circuit		Expended	Total developed	Developed in outer circuit.	
500	0·0	..	0·0	..	..	..	..	..	..	205	.	..	..
,,	·01	·25	·5	..	..	1·63	..	..	3804	,,	·000831	.	
,,	02	·26	·5	..	..	1·63	..	..	·3957	,,	·000864	..	..
,,	·05	·28	1 0	..	..	2·33	..	..	6091	,,	·00190	..	..
,,	·10	·34	1·5	..	..	2 86	..	..	·9079	,,	·00348	..	..
,,	·25	·48	2·0	..	..	3·30	..	..	1·479	,,	·00654	..	..
,,	·50	·72	14·0	1·0	12	8·80	2·33	8·08	5·915	·714	·0698	0409	5·73
,,	·75	·94	180 0	1·5	115	26·58	2 86	25·01	2[illegible]·32	1·48	·881	·587	41·04
,,	1 0	1 16	180 0	2 0	145	31·30	3 30	28 08	33 90	1·83	1 422	987	53 93
,,	1·25	1·36	145 0	2 0	125	28·08	3 30	26·07	35·65	,,	1·342	1·063	58 08
,,	1·5	1 56	135·0	3·0	115	27 09	4 04	25 01	39 46	,,	1·433	1·17	63·93
,,	1·75	1·75	132·0	3·0	102	26·79	4 04	23 55	43·77	,,	1 592	1 21	66·12
,,	2·0	1 94	120·0	3 5	95	25·55	4·36	22·73	46 28	1 73	1 585	1·29	69 94
,,	∞	..	5 0	4·0	0	5 21	4·66	..	..	·714	..	..	..

No. 2.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.

Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance (connected parallel to outer circuit).

Revolutions per minute.	Resistance in S. U.		Reading on electro dynamometer.			Current in webers			E. M. F. in helix in volts	Horse-power.			Percentage of energy turned into useful work
	In outer circuit	Total in circuit	Helix.	Electro-magnet.	Outer circuit	Helix.	Electro-magnet.	Outer circuit.		Expended	Total developed.	Developed in outer circuit.	
600	0·0	24	0	..	..	..	..	..	..	245	..	..	..
,,	·25	·48	1	..	..	2·33	..	..	1·044	,,	·0032	..	..
,,	·50	72	5	..	..	5·21	..	..	3 502	,,	·0244	..	.
,,	·75	·94	125	..	115	26 07	..	25 01	22 87	1 714	·799	·587	34 24
,,	1·0	1·16	215	2·0	175	34·20	3·3	30·86	37·04	2·45	1·70	1 16	47·34
,,	1·25	1·36	195	2 5	153	32 57	3 69	28 84	41·35	,,	1 805	1 3	53 06
,,	1·50	1·56	185	3·5	144	31·72	4·361	27 99	46·20	2·33	1·96	1·47	63 09
,,	1·75	1 75	164	4·0	125	29·87	4 66	26 07	48·80	2·20	1·95	1·49	67 72
,,	2·0	1·94	160	4·0	120	29 50	4·66	25·55	53 43	,,	2 12	1 638	74 22
,,	∞	11·50	6	5·0	0	5·71	5·21	..	65·66	1·10	·503	..	..

No. 3.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance (connected parallel to outer circuit).

Revolutions per minute.	Resistance in S U.		Reading on electro-dynamometer.			Current in webers.			E M F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit	Helix.	Electro-magnet.	Outer circuit	Helix.	Electro-magnet	Outer circuit.		Expended.	Total developed.	Developed in outer circuit.	
700	0 0	·21	..	..	..	..	..	..	..	·29	..	..	..
,,	25	·48	4	..	..	4·66	..	..	2 078	,,	·0129	..	..
,,	·5	·72	11	..	..	7·74	.	..	5 203	,,	·0540	..	.
,,	75	94	285	5	255	39 37	5 21	37 25	34 56	2 57	1 82	1·3	50·58
,,	1 0	1 18	325	6	280	42 05	5 71	39 03	46 33	3 47	2 61	1 9	54 75
,,	1 25	1 36	300	6	250	46 40	5 71	36 88	51 3	,,	2 77	2 18	61·37
,,	1 5	1 56	276	6	205	38 75	5·71	33·19	55·81	,,	2 90	2 09	60 23
,,	1 75	1·75	230	6	170	35 37	5 71	30·41	57·79	,,	2 74	2·03	58·50
,,	2 0	1 94	215	7	158	34 2	6 17	29·31	61 95	3 28	2 84	2 15	65 54
,,	∞	11 50	11	10	0	7·74	7·37	..	83·11	1·29	862	..	..

No. 4.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance (connected parallel to outer circuit).

Revolutions per minute.	Resistance in S U.		Reading on electro-dynamometer.			Current in webers.			E M F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circu't	Helix	Electro-magnet	Outer circuit	Helix	Electro-magnet	Outer circuit		Expended.	Total developed	Developed in outer circuit.	
758	12 30	6·11	38	9 0	9	14 38	6 99	6 99	82 03	2 78	1·58	752	27 26
756	8 90	5 21	49	8 5	14	16·32	6 77	8 88	79 38	2 93	1 74	·878	29 96
734	6 00	4 15	69	9 0	28	19·37	6·99	12 34	75 05	3·0	1 95	1·14	38 00
758	6 00	4·15	72	9·0	28	19 79	6 99	12 34	76 67	3 09	2·03	1·14	36 89
750	4 50	3 13	85	9 0	34	21 50	6 99	13 61	68 85	3·21	1·98	1 04	32·40
756	3 50	2 91	150	7 0	87	28·56	6 17	21·75	77 59	3 86	2 97	2·17	52·07
760	3 00	2·61	185	6·0	107	31 72	5 71	24 12	77·30	3 88	3·29	2·18	56·18
758	2 50	2 28	204	5 0	130	33 31	5 21	26 58	70·91	3 87	3·17	2 21	57 11
750	2 25	2 12	230	..	150	35·37	..	28 56	70 01	3 88	3 32	2 30	60·01
760	2 00	1 94	226	..	157	35 06	..	29 27	63 51	3 88	2 99	2·14	55·15
755	1 75	1 75	252	..	180	37 03	..	31 30	60·50	3·85	3·00	2·14	55·58
754	1 50	1·56	255	..	185	37 25	.	31 72	54 26	3 85	2 71	1 89	49·10
760	1 35	1 44	305	..	240	40·73	..	36·14	54 71	3·72	2 99	2·21	59·41
756	1 25	1·36	310	..	245	41 06	..	36 50	52·14	3·70	2 89	2 08	56·48
764	1 10	1·24	315	..	260	41 20	..	37·61	47 70	3 74	2 63	1·94	51 87
766	1 00	1·16	315	..	264	41 20	..	37 89	44·62	3·59	2·46	1 80	50·01
750	·90	1 07	285	..	238	39·37	..	35·98	39·33	3 06	2 07	1 46	47 71
765	·80	·99	235	..	205	35 75	..	33 39	33 04	2·50	1·58	1 12	44·80
760	·75	·94	208	..	185	33·64	..	31 30	29 52	2 48	1 33	·92	37·09
755	65	85	98	.	87	23 09	..	21 75	18 32	1·08	·567	·385	35·64
760	60	·81	50	.	45	16 49	..	15·64	12 47	1·08	·275	·184	17·03
700	·55	·76	30	..	25	12·77	..	11 68	9·07	·43	·155	094	21·86
752	50	·72	12	..	10	8 07	..	7·37	6 36	·46	·069	·034	7·39
..	·40	·65	5	..	4	5·21	..	4·66	3 16	..	·0221	·0109	2·87
..	∞	11·50	12	12·0	..	8 07	8 07	..	86 64	..	·937	..	..

No. 5.

Helix: 24 part commutator, 336 convolutions, 2·5 m.m. wire, ·4014 S. U. resistance.
Electro-magnets: 512 convolutions, 5·5 m.m. wire, ·3065 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.	Current in webers	E. M. F. in volts	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit				Expended.	Total developed.	Developed in outer circuit.	
750	∞	..	..	..	..	·306	..	..	..
"	8·9	9 61	1	2 33	20·90	"	0653	0605	19·77
"	6 0	6·71	1	"	14 59	"	0455	0408	13 33
"	4 5	5·21	1	"	11 33	"	·0353	0.306	10 00
745	3 5	4·21	7	6 17	24·25	·304	205	167	54 93
760	3 25	3 96	16	9 32	34 46	·620	430	353	56 90
758	3·0	3·71	26	11 89	41·18	·773	656	·531	68 70
750	2·75	3 46	40	14·75	47 65	1·224	·942	·749	61·19
740	2·5	3 20	66	18 95	56 62	1 660	1·44	1 12	67 67
756	2·25	2·96	85	21 50	59 42	2 16	1 71	1 30	60 18
774	2 0	2 71	120	25·55	64 65	2 69	2 21	1 63	60 59
768	1 75	2 46	165	29 96	68 81	3 45	2·76	1 97	57·10
736	1·5	2·21	215	34 20	70 57	3 90	3 23	2 20	56 41
710	1 25	1 96	260	37 61	68 80	4 35	3·47	2 21	50 80
724	1·0	1·71	330	42 37	67 65	5 02	3 84	2 25	44 82
786	9	1·61	520	53 17	79 92	6 16	5 69	3 16	51 29
"	·8	1 51	570	55 67	78·49	6 91	5 86	3 10	44 87
"	·75	1·46	625	58 31	79 48	6 91	6 21	3 19	46 16

No. 6.

Helix: 24 part commutator, 336 convolutions, 2·5 m.m. wire, ·4014 S. U. resistance.
Electro-magnets: 512 convolutions, 5·5 m.m wire, ·3065 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer	Current in webers.	E. M. F. in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit.				Expended	Total developed	Developed in outer circuit.	
500	3 5	..	..	..	.	·257	..	..	..
496	3·0	3 71	1·5	2·86	9 90	·405	·0379	0307	7·58
505	2·5	3·21	2	3·29	9·86	515	1434	·0338	6 563
516	2 0	2·71	9	6 99	17 68	579	165	·122	21 07
520	1·75	2 46	42	15·11	34·70	796	702	·500	62·81
490	1·5	2 21	60	18 06	37 26	1 20	902	612	51·00
496	1·25	1 96	120	25·55	46 75	1·82	1 61	1·02	56 04
490	1·0	1 71	180	31 30	49·97	2·30	2 09	1·23	53 47
490	85	1·56	245	36 50	53·16	2 80	2 60	1 42	50 71
504	·75	1·46	280	39·08	53 20	3·08	2 78	1 43	46 42
502	·65	1·36	320	41 72	52 97	3 08	2 96	1·41	45·79
"	·60	1·31	340	43 01	52 60	3 58	3 03	1 39	38 88
"	·55	1·26	355	43 94	51 68	3 58	3·04	1 33	37 15
488	·50	1·21	385	45·76	51 70	3·58	3 17	1 25	34 92
"	·45	1·16	420	47·80	51·77	3·78	3 32	1·29	34 13
"	·40	1 11	491	51·63	53 49	3·98	3 70	1 33	33 42
"	·35	1·06	510	52 66	52 10	4 08	3·68	1·21	29·65
"	·30	1·01	600	57·12	53 89	4·28	4·12	1·22	28 51
"	·25	·96	630	58·54	52·46	4·58	4 11	1 07	23 36

No. 7.

Helix: 24 part commutator, 336 convolutions, 2·5 m.m. wire, ·4014 S. U. resistance.
Electro-magnets: 512 convolutions, 5·5 m.m. wire, ·3065 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.	Current in webers.	E. M. F. in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit.				Expended.	Total developed.	Developed in outer circuit.	
602	4·5	5·21	..	..	..	·246	..	..	..
„	3·5	4·21	..	..	..	·491	..	..	..
590	3·0	3·71	1	2·33	8·07	·602	·0252	·0208	3·372
602	2·5	3·21	16	9·32	27·93	·676	·349	2725	40·31
606	2·0	2·71	60	18·06	45·69	1·113	1·106	816	73·31
602	1·75	2·46	105	23·90	54·89	1·72	1·760	1·25	72·67
600	1·5	2·21	140	27·59	56·93	2·20	2·110	1·429	64·95
590	1·25	1·96	170	30·41	55·65	2·53	2·27	1·45	57·31
600	1·0	1·71	270	38·32	61·18	3·43	3·14	1·84	53·64
620	·85	1·56	378	45·34	66·04	4·43	4·01	2·18	49·21
„	·75	1·46	400	46·64	63·57	4·68	3·97	2·05	43·80
„	·60	1·31	505	52·41	64·11	5·06	4·50	2·06	40·71

No. 8.

Helix: 24 part commutator, 336 convolutions, 2·5 m.m. wire, ·4014 S. U. resistance.
Electro-magnets: 512 convolutions, 5·5 m.m. wire, ·3065 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.	Current in webers.	E. M. F. in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit.				Expended	Total developed.	Developed in outer circuit.	
715	4·5	5·21	..	..	..	·365	..	..	..
698	3·5	4·21	4	4·66	18·32	·427	·114	·095	22·25
700	3·0	3·71	21	10·69	37·03	1·43	·580	·429	30·00
710	2·5	3·21	50	16·49	49·42	1·59	1·09	·682	42·89
680	2·0	2·71	105	23·91	60·50	2·64	1·94	1·43	54·16
690	1·75	2·46	165	29·96	68·11	3·94	2·76	1·98	50·25
708	1·5	2·21	210	33·80	69·75	3·90	3·16	2·14	54·87
685	1·25	1·96	270	38·32	70·13	4·47	3·60	2·30	51·45
686	1·0	1·71	380	45·46	72·58	5·04	4·42	2·59	51·29
720	·85	1·56	550	54·69	79·66	6·46	5·14	3·18	49·22
„	·75	1·46	620	58·07	79·16	6·76	6·16	3·16	46·74
„	·60	1·31	680	60·83	74·41	7·05	6·07	2·78	39·43
„	·50	1·21	900	69·97	79·05	7·72	7·42	3·06	39·63

No. 9.

Helix: 24 part commutator, 336 convolutions, 2·5 m.m. wire, ·4014 S. U. resistance.
Electro-magnets: 512 convolutions, 5·6 m.m. wire, ·3065 S. U. resistance.

Revolutions per minute.	Resistance in S. U — In outer circuit.	Resistance in S. U — Total in circuit	Reading on electro-dynamometer.	Current in webers	E M F in volts.	Horse power — Expended	Horse power — Total developed	Horse power — Developed in outer circuit	Percentage of energy turned into useful work
450	1	1·71	100	23 33	37 24	1 56	1 16	681	43 65
500	"	"	145	28·08	44·84	2 14	1 69	987	46 12
550	"	"	190	32 14	51 32	2 69	2 21	1 29	47 95
600	"	"	235	35 75	57 08	3·43	2 73	1 60	46 65
650	"	"	290	39 72	63·42	4 24	3 37	1 97	46·46
700	"	"	360	44 25	70 65	5 00	4 18	2 45	49 00
750	"	"	420	47·80	76·32	5 81	4 89	2 86	49 22
800	"	"	490	51·68	82 44	6 86	5 70	3 34	48 69

No. 10.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 3200 convolutions, 2·8 m.m. wire, 7 563 S. U. resistance.

Revolutions per minute	Resistance in S U — In outer circuit	Resistance in S U — Total in circuit	Reading on electro dynamometer — Helix.	Reading on electro dynamometer — Electro-magnet.	Reading on electro dynamometer — Outer circuit.	Current in webers — Helix.	Current in webers — Electro-magnet.	Current in webers — Outer circuit	E M F in helix in volts	Horse-power — Expended.	Horse-power — Total developed	Horse-power — Developed in outer circuit.	Percentage of energy turned into useful work
502	∞	7 8	13	13	..	8 41	8 41	..	61 25	1 33	69	..	..
513	12 0	4 87	30	12	5	12 77	8 08	5 21	58 07	1 47	994	409	27·82
517	10 0	4 54	35	12	7	13 80	8·08	6 17	58 50	1·58	1 082	·476	30 12
520	9 0	4 34	42	12	9	15·11	8 08	7 0	61 23	1 59	1 251	552	34 72
498	7·5	3·99	40	10	10	14 75	7·37	7·37	54 95	1 42	1·086	·510	35·90
501	6 0	3 51	42	8	15	15·11	6 60	9 03	50 36	1 53	1 020	·612	40 00
500	4 5	3 05	50	7	22	16·49	6 17	10 94	46 96	1 53	1 038	·674	44 05
510	3 5	2 62	75	7	35	20 20	6·17	13·8	49·11	1·77	1 338	·834	47 12
502	3 0	2 38	82	6	42	21 12	5 71	15·11	46 93	1 74	1 329	857	49 25
507	2·5	2 11	90	6	50	22 12	5 71	16·49	43 58	1 86	1 292	851	45·75
502	2·0	1 81	120	6	65	25·55	5·71	21·5	43 18	1 95	1 479	1 11	56 92
495	1·75	1 65	140	6	95	27 59	5 71	22 73	42 5	1·92	1 572	1 13	58 85
492	1·5	1·48	160	5	110	29·50	5 21	24 46	40 77	2 01	1 612	1 12	55·72
504	1·25	1·30	190	4	140	32 14	4 66	27·59	39 01	2·16	1 681	1·19	55 09
506	1 1	1 19	205	4	157	33 39	4 66	29·21	37·12	2 27	1 661	1 16	51 10
512	1·0	1 11	200	4	155	32 98	4 66	29 03	34 18	2·09	1·511	1·05	50 23
505	85	·998	225	4	180	34·98	4·66	31 30	32 60	1·96	1 528	1 04	53 06
510	·75	·917	220	4	183	34 60	4 66	31 55	29·63	1 87	1·374	934	49 95
507	·6	·790	160	3	120	29 50	4·04	25·55	21 77	1 45	·861	·490	33 79

No. 11.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 3200 convolutions, 2·8 m.m. wire, 7·563 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro dynamometer.			Current in webers			E. M. F. in helix in volts.	Horse-power			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit	Helix	Electro-magnet.	Outer circuit	Helix.	Electro magnet	Outer circuit		Ex-pended.	Total developed	Developed in outer circuit.	
598	·3	·52	3	1	3	4 04	2 33	4 04	1 962	·24	0106	00613	2 554
596	4	61	4	1	3	4 66	2 33	4 04	2 655	·24	·0166	·00817	3 404
612	·5	70	30	1	26	12 77	2 33	11·59	8 34	375	·143	·0884	28·37
,,	6	·79	180	2		31 30	3·30	..	23 08	87	968	..	.
620	6	·79	215	2	180	34 20	3·30	31·30	25·23	1 77	1 16	·736	41·58
615	·7	·87	280	3	240	39 03	4 04	36·14	31 71	2 26	1·66	1 14	50·44
,,	·75	·917	300	4	250	40·40	4 66	36·38	34·59	2 51	1·87	1·28	51 50
600	·85	998	290	4	245	39 72	4 66	36 50	37 02	2 571	1·97	1·42	55 25
612	1·0	1·11	305	6	235	40 73	5·71	35 75	42·21	2 87	2 30	1·60	55 74
,,	1 25	1·30	290	7	210	39 72	6 17	33 80	48 22	2 86	2 56	1·80	62·93
598	1 5	1 48	230	8	160	35 37	6 60	29·50	48 88	2 81	2·32	1 63	58 00
615	2 0	1 81	195	10	120	32 57	7 34	25 55	55 05	2·89	2 40	1 63	56 40
620	3 0	2 38	130	12	65	26 38	8·08	18 80	59 07	2 66	2 10	1·33	50 00
625	4 5	3·05	85	14	32	21·50	8 88	13 19	61·23	2 30	1 76	98	42 61
,,	6 0	3 59	70	15	22	19·51	9 08	10·94	65·40	2 16	1 71	·898	41 57
620	7 5	3·99	55	16	16	17·29	9·33	9 33	64·42	1·89	1·49	·817	42 22

No. 12.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 3200 convolutions, 2 8 m.m. wire, 7·563 S. U. resistance.

Revolutions per minute	Resistance in S. U		Reading on electro dynamometer.			Current in webers			E. M. F. in helix in volts.	Horse power			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit.	Helix	Electro-magnet	Outer circuit.	Helix	Electro-magnet.	Outer circuit.		Ex-pended	Total developed	Developed in outer circuit	
700	7·5	4 01	80	18	18	20·36	9 89	9 39	78·10	2 57	2·18	·918	35·72
,,	6 0	3·59	80	17	25	20 86	9·62	11 68	69·92	2 57	1 95	1·02	39·69
707	4·5	3 06	95	16	35	22·73	9 33	13 80	61·94	2 88	1 97	1 07	37 15
710	3 0	2·39	165	14	90	29·96	8·88	22 12	66·85	3 19	2 68	1·84	57·87
704	2·5	2 12	195	12	115	32 57	8 08	25 01	64·47	3 59	2 81	1 96	54 59
716	2·0	1 82	235	9	150	35 75	7 0	28·56	60·75	3 94	2 91	2 04	51·77
723	1 5	1·49	315	7	220	41 40	6·17	34 60	57 60	4·28	3 20	2·25	52·57
715	1 25	1 31	350	6	260	43·64	5·71	37 61	53·38	4 23	3·12	2 21	52 25
722	1·0	1·12	385	5	300	45·76	5·21	40·40	47·85	4 27	2 93	2·04	47·77
705	75	·92	370	5	305	44 86	5·21	40·73	38·53	3 45	2 31	1 56	45 21
703	·60	·80	300	5	270	40 40	5·21	38·32	30 18	2 29	1 63	1 10	48·03
716	·50	·71	150	4	100	28·56	4·66	23·32	18·94	·292	·725	·34	11·64
710	·40	·62	7	1	7	6 17	2·33	6·17	3·571	290	·295	·019	6 552

No. 13.

Helix: 24 part commutator, 312 convolutions, 2·8 m m. wire, ·234 S. U. resistance.
Electro-magnets: 3200 convolutions, 2·8 m.m. wire, 7·563 S. U. resistance.

Revolutions per minute	Resistance in S. U.		Reading on electro-dynamometer			Current in webers.			E M F in helix in volts	Horse power,			Percentage of energy turned into useful work
	In outer circuit.	Total in circuit.	Helix.	Electro-magnet	Outer circuit	Helix	Electro magnet	Outer circuit		Ex-pended	Total developed	Developed in outer circuit.	
450	1·0	1 12	140	..	105	27 59	..	23 91	28 85	1·47	1 02	·715	48 63
500	,,	,,	185	..	150	31 72	..	28 56	33 16	1 94	1 41	1 02	52 57
550	,,	,,	250	..	195	36 88	..	32 57	38 56	2 47	1 91	1 33	53 85
600	,,	,,	325	..	260	42 05	..	37 61	43 97	3 06	2 48	1 77	57 84
650	,,	,,	400	..	305	46 64	..	40 73	48 77	3 71	3 05	2·08	56 06
700	,,	,,	470	..	350	50 56	..	43 64	52 87	4 43	3 58	2 38	53 72
750	,,	,,	530	..	395	53 69	..	46 35	56 14	5 20	3 91	2 69	51 73
800	,,	,,	640	..	460	58 98	..	50 03	61 69	5 88	4 86	3 13	53 23
850	,,	,,	700	..	510	61 69	..	52 66	64 51	6 42	5 33	3 17	54 05
900	,,	,,	785	..	570	65 33	..	55 67	68 31	6 98	5 98	3·88	55 59
850	,,	,,	700	..	505	61 69	..	52 41	64 51	6 42	5 33	3 44	53 58
800	,,	,,	600	..	440	57 12	..	48 92	59 73	5 71	4 57	2 99	52 36
750	,,	,,	510	..	372	52 66	..	44 93	55 07	4 90	3 80	2 53	51 63
700	,,	,,	440	..	330	48 92	..	42 37	51 15	4 14	3 35	2 25	54 35
650	,,	,,	350	..	270	43 64	..	38 32	45 63	3 45	2 67	1 84	53 33
600	,,	,,	285	..	220	39 47	..	34 60	41 27	2 82	2 18	1 5	53 19
550	,,	,,	240	..	180	36·14	..	31 30	37 79	2 24	1 83	1 23	54 91
500	,,	,,	180	..	135	31 30	..	27 09	32 78	1 78	1 37	918	53 06
450	,,	,,	135	..	100	27 09	..	23 32	28 33	1·28	1 03	681	53 20

No. 14.

Helix: 24 part commutator, 312 convolutions, 2 8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 2240 convolutions, 3 m m wire, 4·46 S. U resistance.

Revolutions per minute.	Resistance in S U		Reading on electro dynamometer			Current in webers			E M F in helix in volts.	Horse power.			Percentage of energy turned into useful work
	In outer circuit,	Total in circuit,	Helix	Electro magnet	Outer circuit	Helix	Electro magnet	Outer circuit		Ex-pended	Total developed.	Developed in outer circuit.	
505	∞	4 70	23	23	..	11 19	11 19		49 01	1 34	·736		.
510	7 0	2 96	55	..	9	17 29	..	6 90	47 79	1 77	1 11	428	24 18
502	6 0	2 8	60	..	12	18 06	..	8 08	47 21	1 74	1 14	490	28 16
502	5 0	2 6	65	18	14	18 80	9 89	8 88	45 64	1 74	1 15	493	28 33
492	4 5	2 48	65	16	16	18 80	9 33	9 32	43 53	1 71	1 10	489	28 60
,,	3 0	2 03	120	15	40	25 55	9 08	14 75	48·43	1 71	1 66	817	47·78
515	2 5	1 84	125	14	53	26 07	8 88	16 98	44 79	1 99	1 57	·902	45 32
494	2·0	1 62	145	13	7	28 08	8 40	19 51	42 47	1 91	1 60	953	49 39
,,	1 5	1 36	180	10	100	31·30	7 37	23 32	39 75	2 01	1 67	1 02	50·74
508	1 25	1 22	210	.	130	33 80	.	26 58	38 50	2 18	1 74	1·10	50 46
512	1 0	1 06	260	9	170	37·61	6 90	30 41	32 73	2·40	1 65	1 16	48·33
515	1 0	1 06	280	..	180	39 03	..	31 30	38 65	2 41	2 02	1 23	51 03
506	9	99	295	.	205	40 06		33 39	36 93	2 37	1 98	1 25	52 74
515	8	·92	220	7	220	34 60	6 17	34 60	29 72	2 52	1 38	1·20	47 61
,,	·7	84	295	..	222	40 06	..	34 75	31 42	2 52	1 69	1·05	41 67
498	·3	·52	95	2	75	22·73	3·29	20 20	11 04	0 71	·336	153	21 55
510	·6	77	280	5	200	39 03	5 21	32 98	28 06	2 18	1 47	817	37 48
,,	·5	·69	275	..	210	38 68	..	33·80	24 92	2 42	1·29	715	29 54

No. 15.

Helix: 24 part commutator, 312 convolutions, 2·8 m.m. wire, ·234 S. U. resistance.
Electro-magnets: 2240 convolutions, 3 m.m. wire, 4·46 S. U. resistance.

Revolutions per minute	Resistance in S U		Reading on electro dynamometer.			Current in webers.			E M F in helix in volts.	Horse power.			Percentage of energy turned into useful work
	In outer circuit	Total in circuit	Helix	Electro magnet.	Outer circuit.	Helix	Electro-magnet	Outer circuit		Ex-pended	Total developed	Developed in outer circuit	
700	..	·24	..	..	..	..	..	..	..	·28	..	..	..
"	·1	31	2	..		3 29	.	..	1·044	·28	0046	..	..
"	2	·43	3	..	.	4·03	..	.	1 617	·28	·0087		.
712	·3	52	14	..	10	8 83	..	7 37	4 311	29	0513	0204	7 034
698	4	61	15	..	12	9 03	.	8 07	5 143	28	·0622	0326	11 64
712	·5	·69	370	4	305	44 86	4 66	40 73	28·89	2·61	1 74	1 04	39 84
715	6	77	510	6	370	52 66	5 71	41 86	37 86	3 50	2·67	1 51	43·14
"	·75	88	570	.	360	55 67		44 25	45 74	4 08	3 41	1 83	44 85
720	1 0	1 06	630	15	335	53 69	9 03	42 69	53 14	4 10	3 82	2 28	55 61
714	1 25	1 22	440	18	260	48 93	9 89	37 61	55 73	4 08	3 66	2 21	54 16
708	1·5	1·36	410	21	215	47 23	10 69	34 20	59 97	4 05	3 80	2 20	54 32
700	2 0	1 62	325	25	150	42 05	11 68	28 56	63·60	4 00	3 58	2 04	51·00
712	3 0	2 03	230	32	80	35 37	13 19	20 86	67 04	3 48	3 18	1 63	46·83
"	∞	4 7	..	50	..	..	16·65	..	..	2 47	..	..	

No. 16.

Helix: 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets: 3916 convolutions, 2·5 m m. wire, 11 26 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E M F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit	Helix.	Electro-magnet	Outer circuit	Helix	Electro-magnet	Outer circuit		Ex-pended	Total developed	Developed in outer circuit	
612	0 0	..	..	..		.	.	..	..	25	..	..	..
640	25	418	..	..	..	.	..	..	..	25	..	..	..
625	5	652	4	..	..	4 66	..	..	2 84	·25	·018		.
620	75	876	97	..	90 0	22 97	.	22 12	18 8	·76	·579	·46	60 6
630	1 0	1 09	220	2 0	190·0	34 6	3 3	32 14	35 21	2·44	1 63	1 29	52 86
630	1 25	1 29	225	3 0	190 0	34 98	4·04	32·14	41 13	2 57	1·93	1 61	62 64
635	1 5	1 5	205	3·5	170 0	33 39	4 36	30 41	46 76	2 59	2 09	1 73	66·79
636	1 75	1 69	185	3·5	145 0	31 72	4 36	28 08	50 05	2 46	2 18	1·73	70 32
638	2 0	1 87	165	4 0	120 0	29 96	4 66	25 55	52·3	2 47	2 10	1 63	65 99
628	4 5	3 38	68	6 0	32 0	19 23	5 71	13 19	60·69	2 05	1·56	·98	47·86
621	6 5	4 29	42	7 0	15 0	15 11	6 16	9 03	60 52	1 91	1 22	66	34 55
635	9 0	5 17	32	7 5	8 5	13 19	6 39	6 8	63 67	1·81	1 13	521	28·78
610	11 0	5 73	25	7 0	7 0	11 68	6 16	6 16	62 49	1 62	98	·522	32 22
614	∞	11 43	8	8 0	..	6 6	6 6	..	70 43	1·25	623	..	..

No. 17.

Helix: 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E. M. F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit	Helix.	Electro-magnet.	Outer circuit.	Helix.	Electro-magnet.	Outer circuit.		Ex-pended.	Total developed.	Developed in outer circuit.	
665	·25	·42	3	..	3	4·04	..	4·04	1·58	·271	·0085	·0051	1·82
670	·5	·652	30	..	28	12·77	..	12·34	7·74	·547	·132	·095	17·37
665	·75	·875	245	2·0	225	36·50	3·3	34·98	29·85	2·01	1·45	1·14	55·88
660	·85	·968	255	2·5	235	37·25	3·69	35·75	33·49	2·29	1·67	1·36	59·38
640	1·0	1·09	290	3·0	260	39·72	4·04	37·61	40·42	2·61	2·15	1·77	67·81
645	1·1	1·17	272	3·0	230	38·47	4·04	35·37	42·02	2·63	2·17	1·72	65·40
650	1·25	1·29	250	3·5	205	36·88	4·36	33·39	44·42	2·55	2·2	1·71	65·66
645	1·4	1·42	235	4·0	185	35·73	4·66	31·72	47·39	2·63	2·27	1·76	66·92
650	1·5	1·5	228	4·0	180	35·22	4·66	31·3	49·33	2·52	2·33	1·84	73·01
660	1·75	1·69	200	4·5	153	32·98	4·94	28·81	52·01	2·56	2·3	1·82	71·09
660	2·0	1·87	180	5·0	145	31·3	5·21	28·08	51·65	2·42	2·29	1·97	81·40

No. 18.

Helix: 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets: 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E. M. F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit.	Helix.	Electro-magnet.	Outer circuit.	Helix.	Electro magnet.	Outer circuit.		Ex-pended.	Total developed.	Developed in outer circuit.	
817	1·1	1·17	435	..	360	48·65	..	44·25	53·15	4·67	3·47	2·69	57·6
812	1·0	1·09	465	..	385	50·29	..	45·76	51·11	4·81	3·44	2·61	54·28
815	1·25	1·29	405	5·0	385	46·94	5·21	42·69	56·53	4·82	3·56	2·85	59·12
820	1·5	1·5	370	5·5	290	44·86	5·47	39·72	62·83	4·85	3·78	2·96	61·03
814	1·75	1·69	330	..	250	42·37	..	36·88	66·85	4·82	3·8	2·98	61·82
812	2·0	1·87	309	..	240	40·40	..	34·6	70·54	4·84	3·82	3·00	61·65
824	·9	1·01	450	..	385	49·47	..	45·76	46·65	4·71	3·09	2·36	50·10
820	·8	·92	450	..	390	49·47	..	46·06	42·56	4·35	2·82	2·12	48·73
808	·75	·876	430	..	390	48·36	..	46·06	39·55	4·12	2·66	1·99	48·30
840	·6	·743	220	..	215	34·60	..	34·2	24·00	2·4	1·11	·878	36·58
830	·5	·652	..	..	100	..	..	23·32	..	2·37	..	·34	14·34
820	·5	·652	75	..	70	20·20	..	19·51	12·30	2·34	·333	·238	10·16

No. 19.

Helix : 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets : 3916 convolutions, 2·5 m.m. wire, 11·26 S. U. resistance.

Revolutions per minute	Resistance in S U.		Reading on electro-dynamometer.			Current in webers.			E M F in helix in volts.	Horse-power.			Percentage of energy turned into useful work
	In outer circuit.	Total in circuit.	Helix	Electro magnet	Outer circuit.	Helix.	Electro-magnet.	Outer circuit.		Ex-pended.	Total developed	Developed in outer circuit.	
480	1 0	1 091	98	..	85	23 09	..	21 5	23 53	1 37	728	578	42 19
530	„	„	135	..	123	27 09	..	25 86	27 60	1 73	1 00	837	48 38
602	„	„	228	..	195	34 88	..	32·57	35·49	2 33	1 66	1 33	57·08
670	„	„	290	..	250	39 72	..	36 88	40 47	3 14	2 15	1·7	54·14
744	„	„	390	..	320	46 06	..	41 72	46 93	3 95	2 89	2 18	55 19
800	„	„	455	..	380	49 75	..	45 46	50·70	4 73	3 38	2 59	54·76
864	„	„	515	..	430	52 92	..	48 36	53·92	5 64	3 82	2 93	51 95
911	„	„	585	..	490	56·42	..	51 63	57 49	6 32	4 35	3 34	52 85

No. 20.

Helix : 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets : 3200 convolutions, 2·8 m m. wire, 7·563 S. U. resistance.

Revolutions per minute.	Resistance in S U.		Reading on electro dynamometer			Current in webers.			E. M. F. in helix in volts.	Horse-power.			Percentage of energy turned into useful work
	Outer circuit	Total in circuit	Helix	Electro-magnet	Outer circuit	Helix	Electro magnet.	Outer circuit		Ex-pended.	Total developed	Developed in outer circuit.	
716	∞	7 74	26	26 0	..	11·89	11 89	..	85 93	2·04	1 37	.	..
710	2 0	1 75	300	15 0	195	40 4	9 03	32 57	66 01	3 62	3 57	2·66	73·48
706	1·75	1 59	320	13 5	220	41 72	8 57	34 6	61 93	3 74	3·46	2 62	70 05
718	1·5	1 42	360	12 0	255	44 25	8 08	37 25	58 66	3·96	3 48	2·61	65 91
710	1 25	1 25	395	9 0	290	46 35	6 99	39 72	54 10	4 06	3·36	2·47	60 84
708	1 0	1 06	445	7 0	340	49 21	6·16	43 01	48 70	4 04	3 21	2 31	57 18
714	95	1 02	460	6 0	360	50 03	5 71	44 25	47 65	3·79	3 2	2·33	61 47
706	9	977	400	..	370	50·03	.	44 86	45 64	3 75	3 19	2 27	60 53
710	8	896	460	5 0	380	50 23	5 21	45 46	41 85	3 62	2 81	2 07	57 18
714	·7	814	445	4·0	375	49 20	4 66	45 16	37 40	3 35	2 47	1 78	53 13
715	6	729	390	..	340	46 06	..	43 01	31·35	2·77	1·94	1 39	50·18
715	5	642		..	230	..	..	35 37	..	.	..	875	.
710	5	·642	205	..	195	33·39	..	32 57	20 02	1 3	896	685	51·15

No. 21.

Helix : 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets : 3200 convolutions, 2·8 m.m. wire, 7·563 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E. M. F in helix in volts.	Horse power.			Percentage of energy turned into useful work.
	Outer circuit.	Total in circuit	Helix.	Electro-magnet.	Outer circuit	Helix.	Electro-magnet.	Outer circuit		Ex-pended.	Total developed.	Developed in outer circuit.	
468	1·0	1·06	170	..	130	30·41	..	26·58	30·10	1·34	1·23	·884	65·97
532	"	"	235	..	185	35·75	..	31 72	35 38	1 96	1·69	1·26	64 21
610	"	"	310	..	240	41·06	..	36 14	40·61	2 61	2 24	1 64	62·83
678	"	"	365	..	290	44·56	..	39·72	44·10	3·32	2·63	1·98	59 64
716	"	"	430	..	335	48·36	..	42·68	47 86	3·79	3 10	2 28	60·16
788	"	"	510	..	385	52·66	..	45 76	52·12	4 35	3·67	2·63	60·46
860	"	"	620	..	430	57 14	..	48·36	56·55	5·08	4 33	2 93	57 68

No. 22.

Helix : 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets : 2556 convolutions, 3 m.m. wire, 4·46 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro-dynamometer.			Current in webers.			E. M. F. in helix in volts.	Horse power.			Percentage of energy turned into useful work.
	In outer circuit	Total in circuit.	Helix.	Electro-magnet.	Outer circuit.	Helix.	Electro-magnet.	Outer circuit.		Ex-pended.	Total developed.	Developed in outer circuit.	
690	∞	4·63	46	46	..	15·82	15·82	..	68·4	2·67	1·45	..	..
702	2·0	1·55	300	24	155	40·4	11·43	29·03	58·47	4 01	3·16	2·11	52·62
694	1 75	1·43	300	22	170	40·4	10·94	30·41	53·94	3·97	2·92	2 03	51 13
698	1·5	1·29	335	20	205	42 69	10·43	33·39	51 42	3 99	2 94	2·09	52·38
696	1·25	1·15	390	18	245	46·06	9·9	36 50	49·46	4·12	3·06	2 08	50·48
696	1·0	·99	395	14	265	46·35	8·88	37 97	42·85	3·98	2·67	1·8	45·22
690	·9	·92	390	9	290	46·06	6 99	39·72	39·56	3·52	2·44	1·78	50·57
649	·8	·85	385	8	290	45·75	6·6	39·72	36·32	3·39	2·22	1·58	46·61
706	·7	·78	360	7	270	44·25	6·16	38·32	32·14	3·02	1 9	1 29	42 71
684	·6	·702	290	..	245	39·72	..	36 50	26 02	2·65	1 38	1·0	37·73
684	·5	·623	..	..	160	..	..	29·50	..	1·26	..	·544	43·18
684	·5	·623	140	..	..	27·59	..	..	16·05	1·23	·594	..	..

No. 23.

Helix: 24 part commutator, 288 convolutions, 3 m.m. wire, ·173 S. U. resistance.
Electro-magnets, 2556 convolutions, 3 m.m. wire, 4·46 S. U. resistance.

Revolutions per minute.	Resistance in S. U.		Reading on electro dynamometer.			Current in webers.			E. M. F. in helix in volts.	Horse power.			Percentage of energy turned into useful work.
	In outer circuit.	Total in circuit	Helix.	Electro-magnet.	Outer circuit	Helix.	Electro-magnet.	Outer circuit.		Ex-pended.	Total developed.	Developed in outer circuit.	
492	1	99	240	7·0	160	36·11	6·16	29·5	33·41	2·11	1 62	1·09	51·66
528	"	"	282	9·0	185	39·17	6·99	31 72	36·21	2·48	1·9	1·26	50 81
602	"	"	345	12·0	235	43·31	8·08	35·75	40·04	3·07	2·32	1·6	52 12
634	"	"	410	14·0	280	47·23	8 88	39·03	43·66	3·61	2·76	1·9	52·63
722	"	,	450	16 0	300	49·47	9·33	40 4	45 73	4·27	3 03	2·04	47 78
790	"	"	540	18 5	335	54·20	10·03	42·69	50·10	5·0	3·64	2·28	45·60
856	"	"	610	21·0	370	57·60	10·69	44·86	53·24	5·76	4·11	2·52	43·74

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TO THE

PHILOSOPHICAL TRANSACTIONS

FOR THE YEAR 1880.

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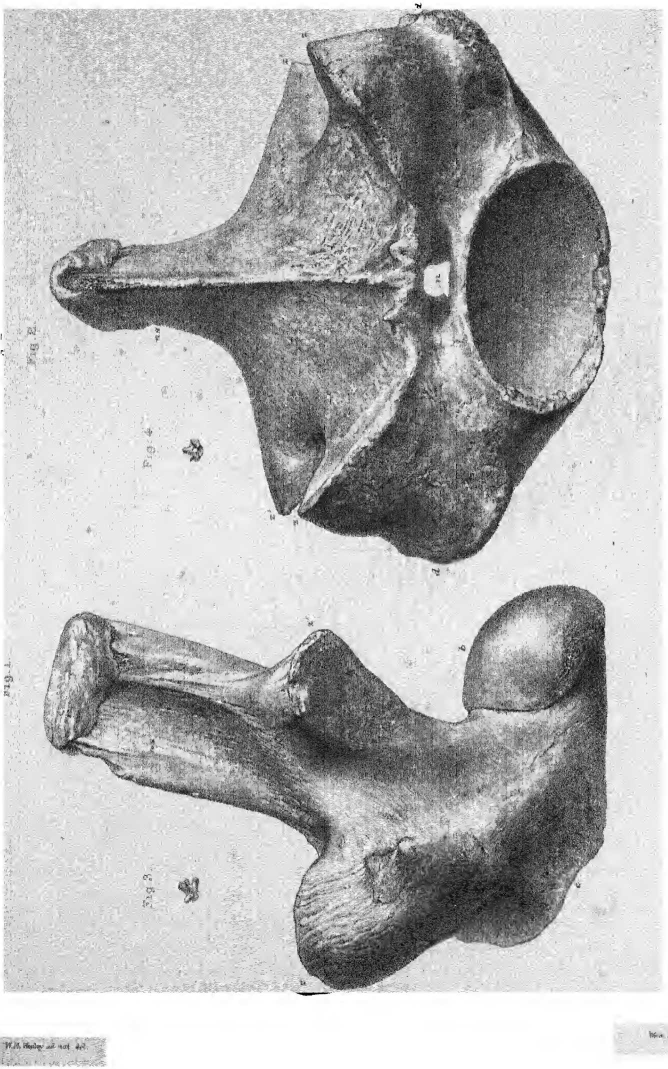

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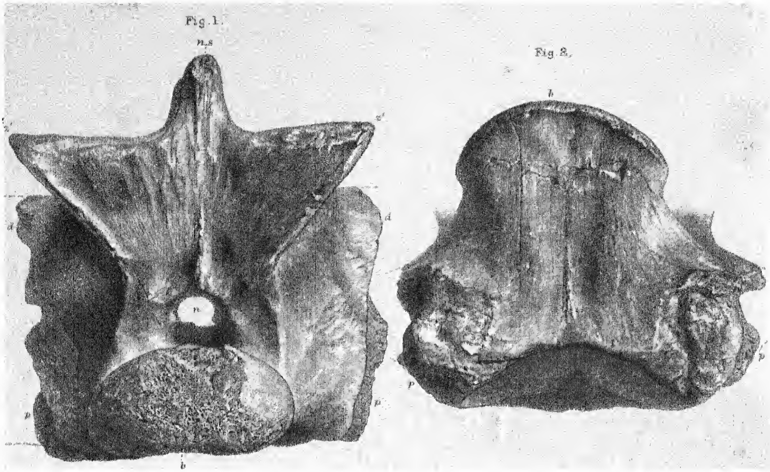
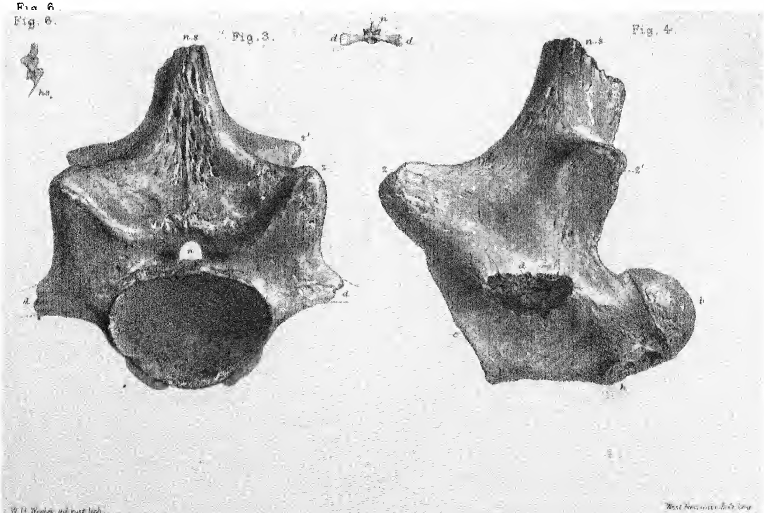
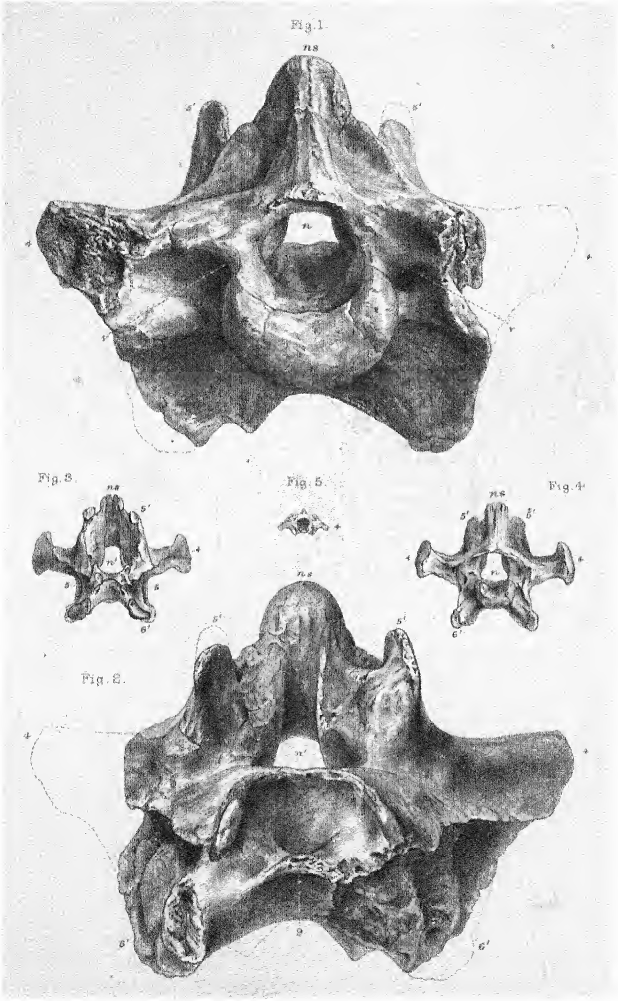


Fig. 5

Fig. 6.

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4'
4'
Fig. 3.
ns
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Fig. 5.
4
Fig. 4.
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Fig. 2.
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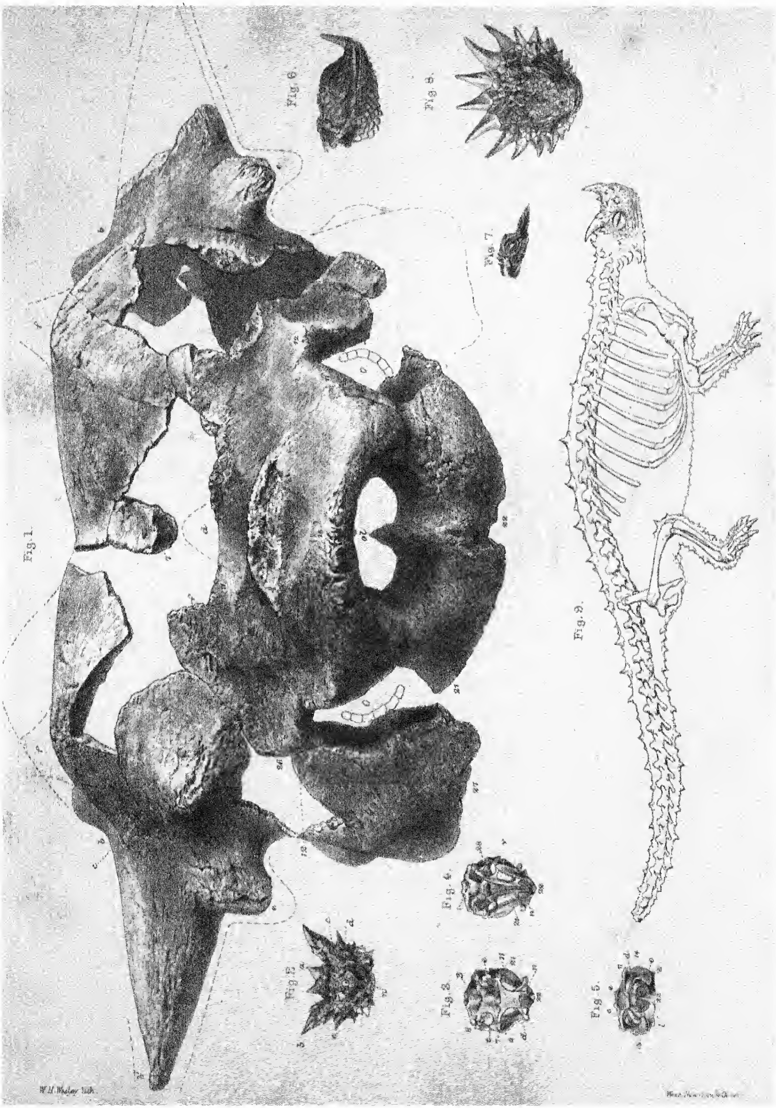
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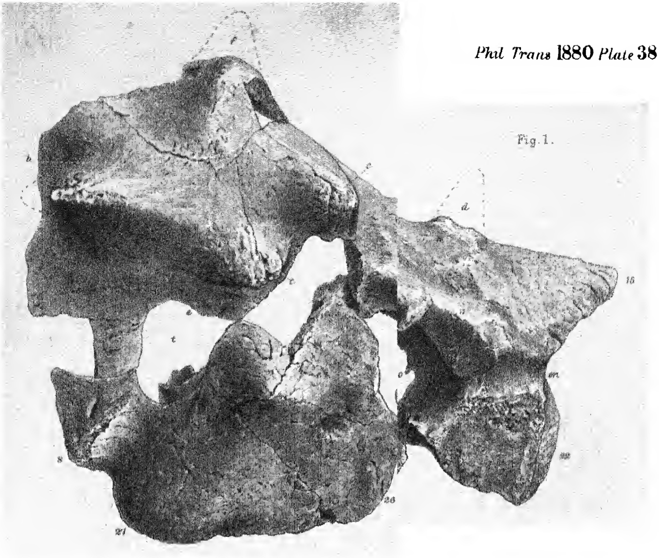
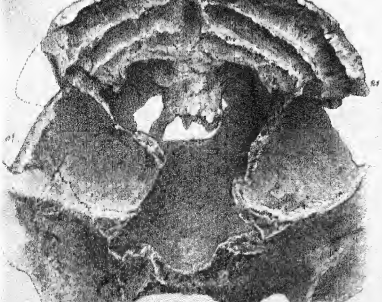


Fig. 1.

Fig. 3.



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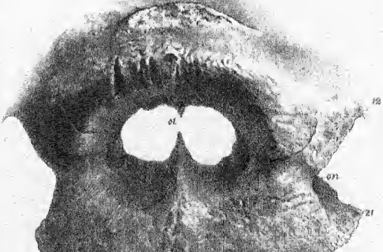

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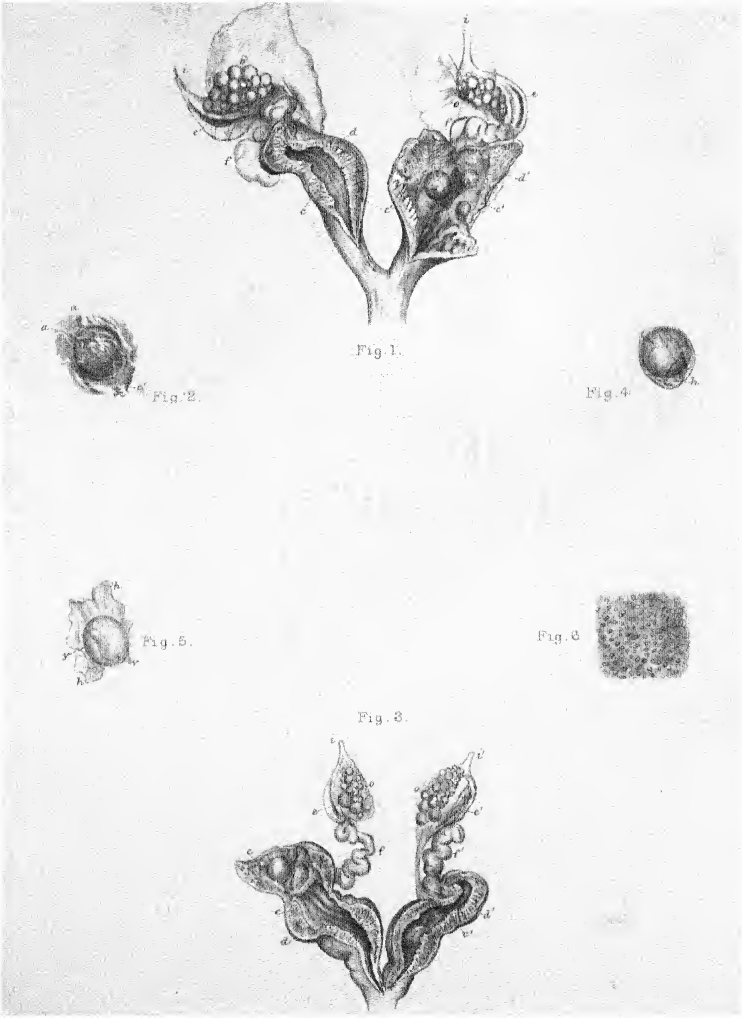
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Fig.5.
Fig.6.
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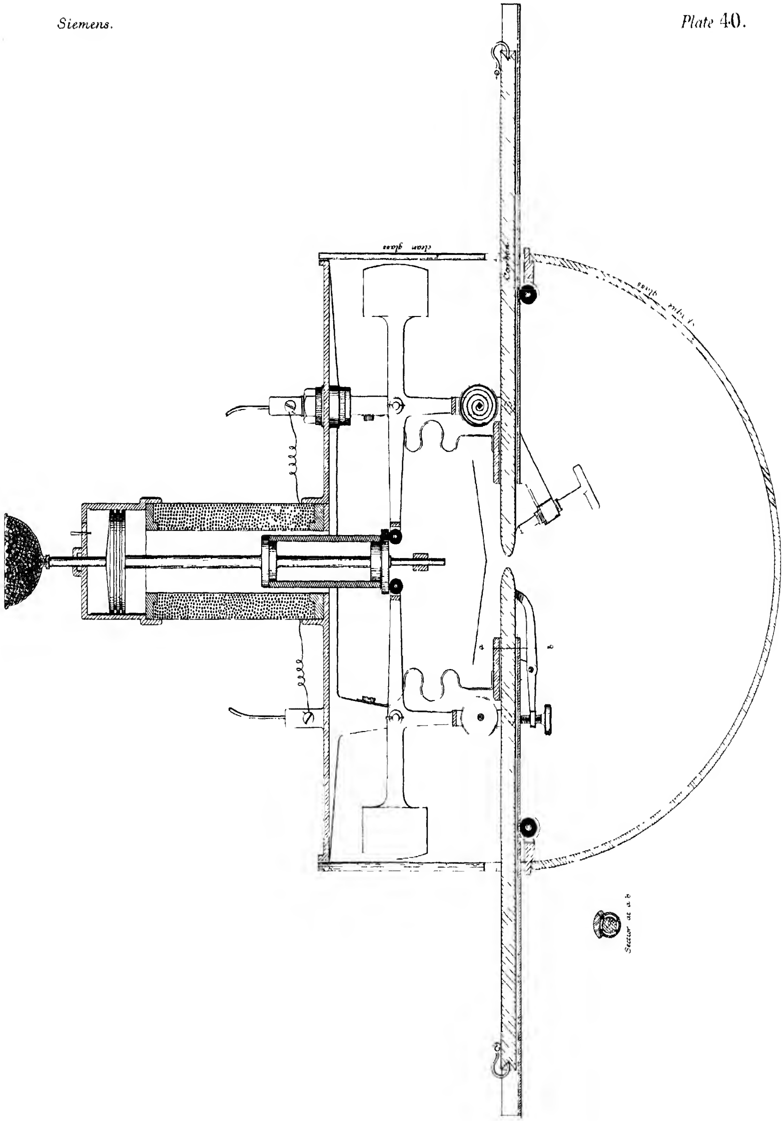
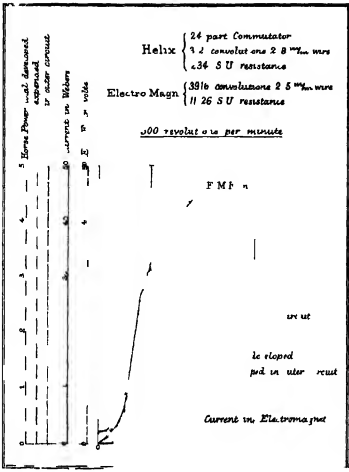
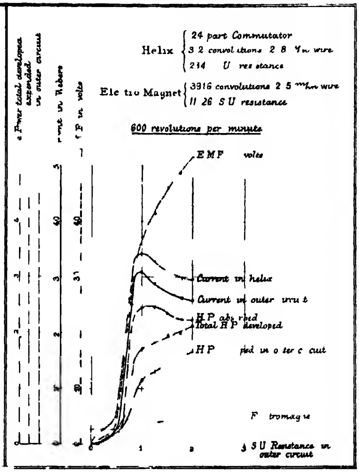

clear glass
Section at a.b

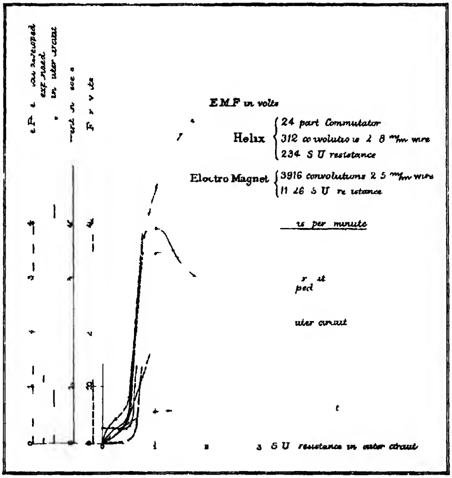
N° 1


Helix
24 part Commutator
234 S U resistance
Electro Magn
3916 convolutions 2 5 m/m wire
11 26 S U resistance
revolutions per minute
Horse Power
Current in Webers
E M F in volts
E M F
Current in Electromagnet

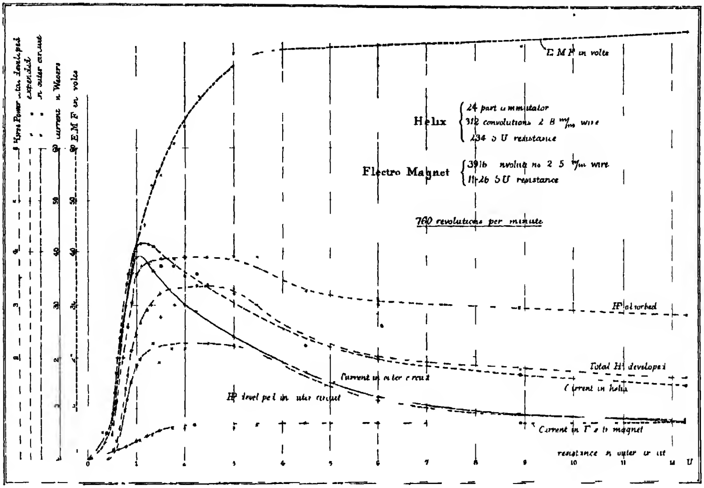
N° 2


Helix
24 part Commutator
234 U resistance
Electro Magnet
3916 convolutions 2 5 m/m wire
11 26 S U resistance
600 revolutions per minute
E M F volts
Current in helix
Current in outer circuit
Total H P developed
H P
S U Resistance in outer circuit

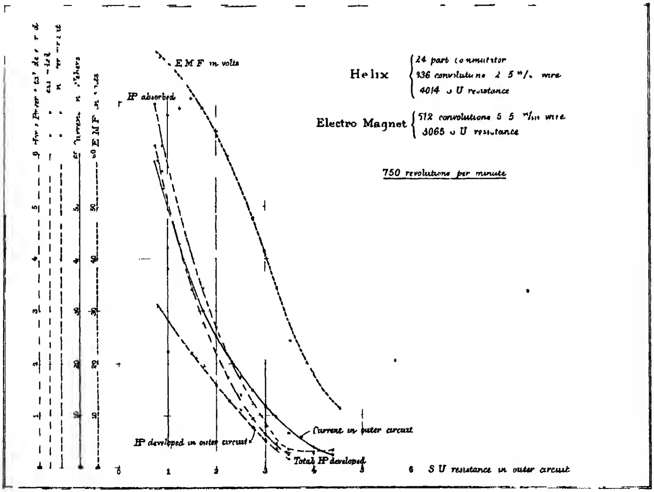
N° 3


E M F in volts
Helix
24 part Commutator
234 S U resistance
Electro Magnet
3916 convolutions 2 5 m/m wire
per minute
uter circuit
S U resistance in outer circuit

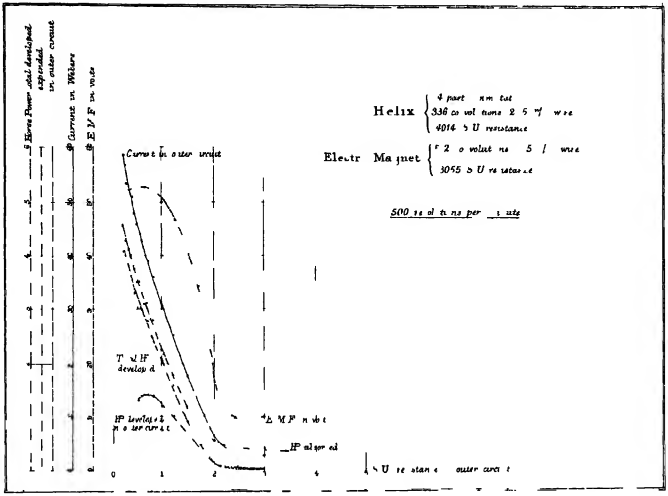
N° 4


E M F in volts
Helix
24 part commutator
312 convolutions 2 8 m/m wire
234 S U resistance
Electro Magnet
391 convolutions 2 5 m/m wire
11.26 S U resistance
760 revolutions per minute
HP absorbed
Total HP developed
Current in helix
Current in outer circuit
HP developed in outer circuit
Current in electro magnet
resistance in outer circuit
EMF in volts

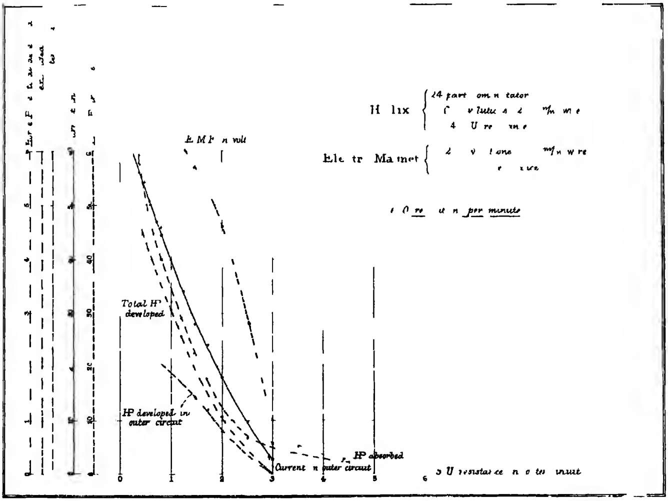
N° 5


E M F in volts
HP absorbed
Helix
24 part commutator
336 convolutions 2 5 m/m wire
4014 S U resistance
Electro Magnet
512 convolutions 5 5 m/m wire
3065 S U resistance
750 revolutions per minute
Current in outer circuit
HP developed in outer circuit
Total HP developed
S U resistance in outer circuit

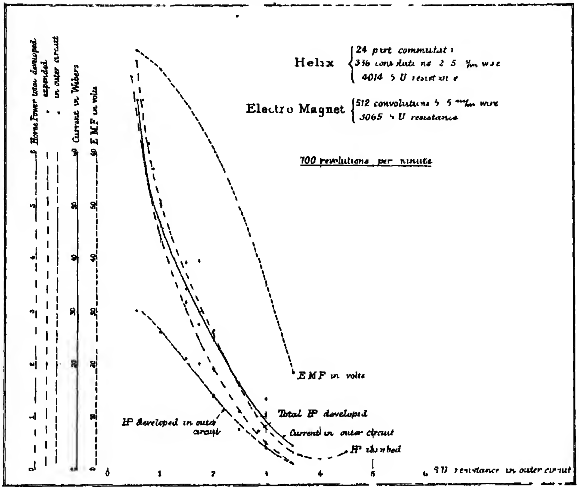
N° 6


Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
Helix
336 co vol tions
4014 S U resistance
Electr Magnet
3055 S U resistance
Current in outer circuit
developed
E M F
HP absorbed
S U resistance outer circuit
0
1
2
4

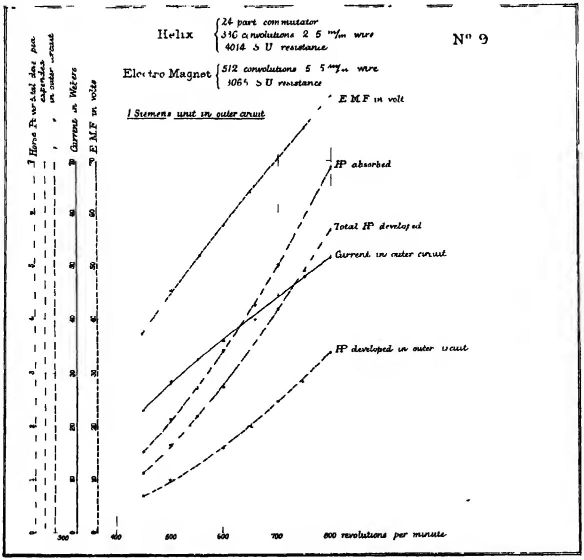
N 7


Helix
24 part commutator
Electr Magnet
per minute
E M F in volt
Total HP developed
HP developed in outer circuit
HP absorbed
Current in outer circuit
S U resistance
0
1
2
3
4
5
6

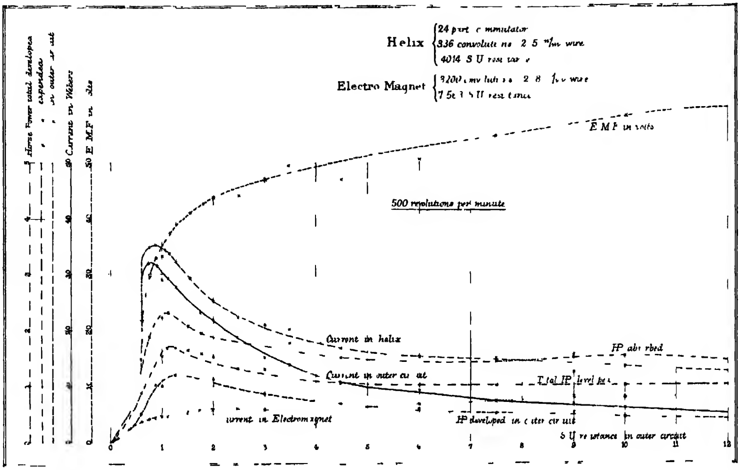
N° 8


Helix
24 part commutator
336 convolutions 2 5 m/m wire
4014 S U resistance
Electro Magnet
512 convolutions 5 5 m/m wire
3065 S U resistance
700 revolutions per minute
Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
E M F in volts
Total HP developed
Current in outer circuit
HP absorbed
HP developed in outer circuit
S U resistance in outer circuit

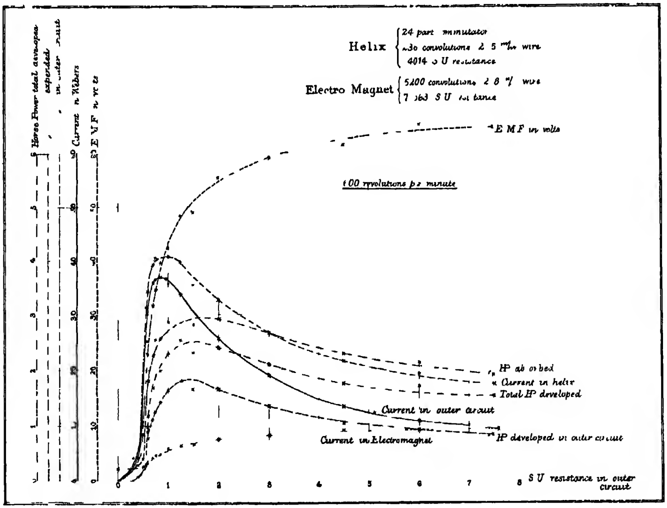
N° 9


Helix
24 part commutator
336 convolutions 2 5 m/m wire
4014 S U resistance
Electro Magnet
512 convolutions 5 5 m/m wire
3065 S U resistance
1 Siemens unit in outer circuit
E M F in volt
HP absorbed
Total HP developed
Current in outer circuit
HP developed in outer circuit
Current in Webers
E M F in volts
300
400
500
600
700
800 revolutions per minute

N° 10

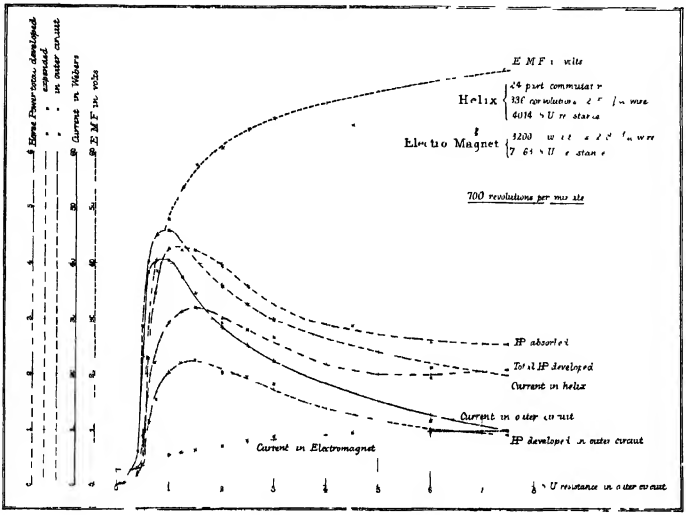


N° 11

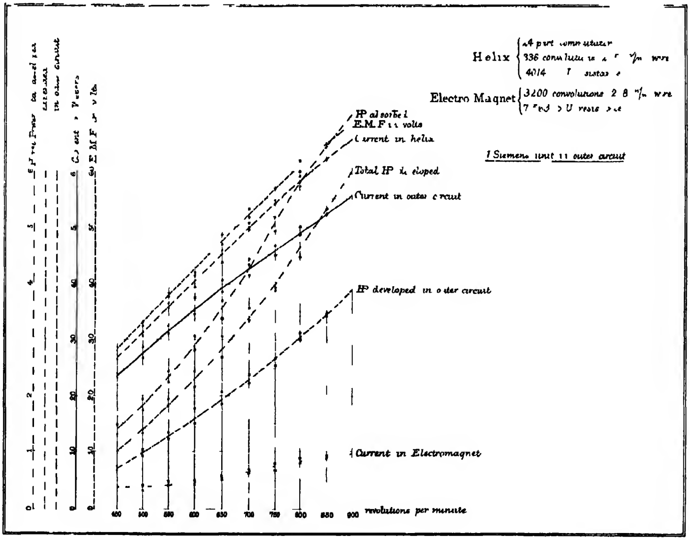


W. Newman & Co. imp.

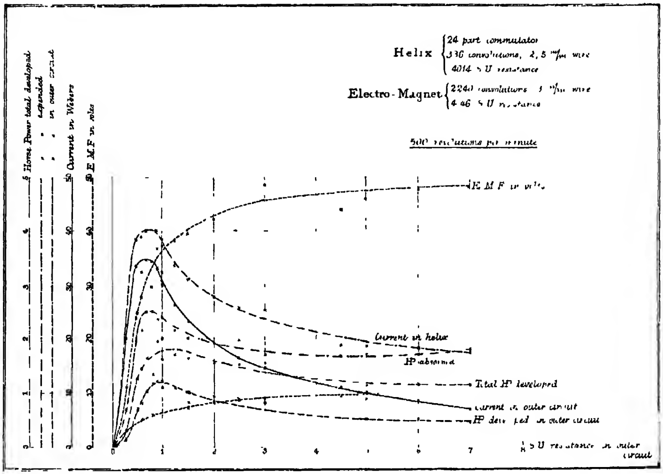
N° 12


Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
Helix
24 part commutator
Electro Magnet
700 revolutions per minute
HP absorbed
Total HP developed
Current in helix
HP developed in outer circuit
Current in Electromagnet
U resistance in outer circuit

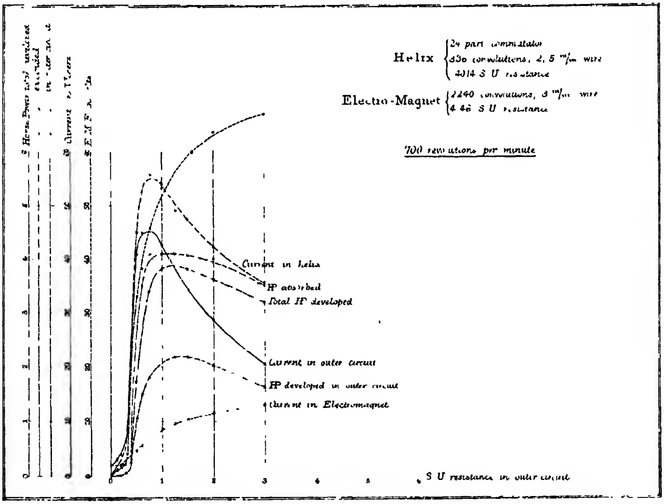
N° 13


Helix
336
4014
Electro Magnet
3200 convolutions
HP absorbed
E.M.F in volts
Current in helix
Total HP developed
Current in outer circuit
1 Siemens unit in outer circuit
HP developed in outer circuit
Current in Electromagnet
revolutions per minute

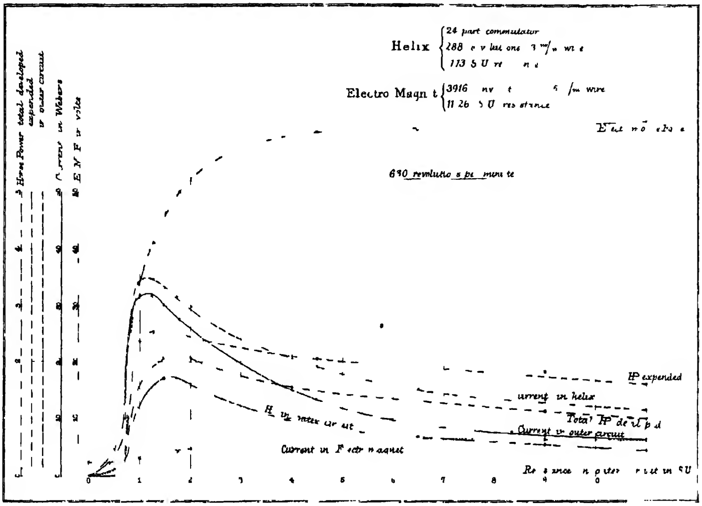
N° 14


Helix
24 part commutator
4014 S U resistance
Electro-Magnet
500 revolutions per minute
E M F in volts
Current in helix
HP absorbed
Total HP developed
current in outer circuit
HP developed in outer circuit
S U resistance in outer circuit
Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts

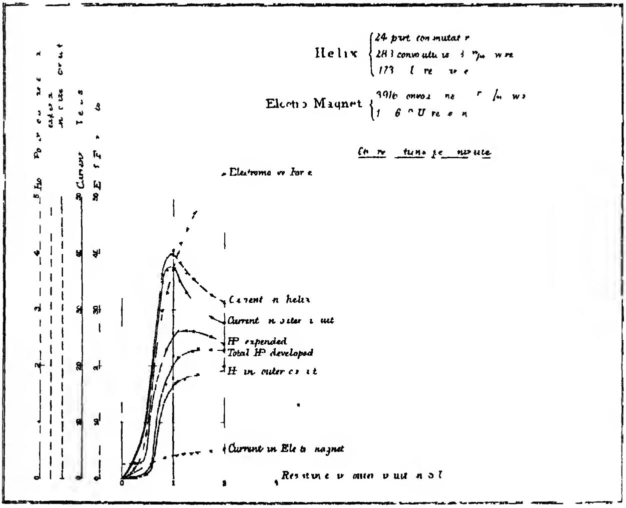
N° 15


Helix
24 part commutator
4014 S U resistance
Electro-Magnet
2240 convolutions, 3 m/m wire
4 46 S U resistance
700 revolutions per minute
Current in helix
HP absorbed
Total HP developed
Current in outer circuit
HP developed in outer circuit
Current in Electromagnet
S U resistance in outer circuit

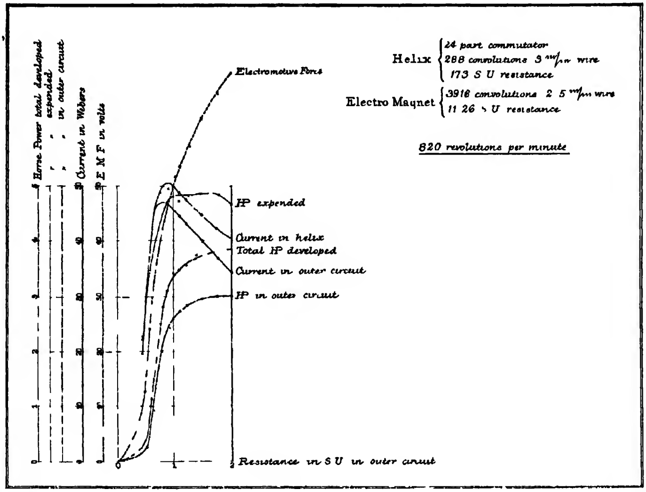
N°16


Helix
24 part commutator
Electro Magnet
HP expended
Current in helix
Current in outer circuit
Current in Electromagnet

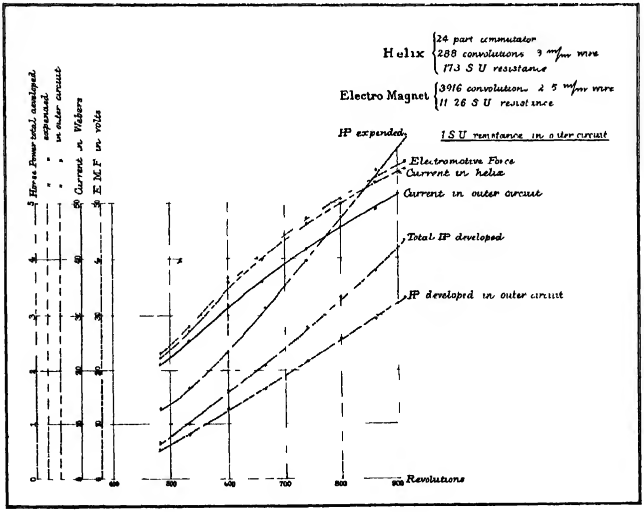
N°17


Helix
24 part commutator
Electro Magnet
Current in helix
HP expended
Total HP developed
Current in Electromagnet

N°18



N°19



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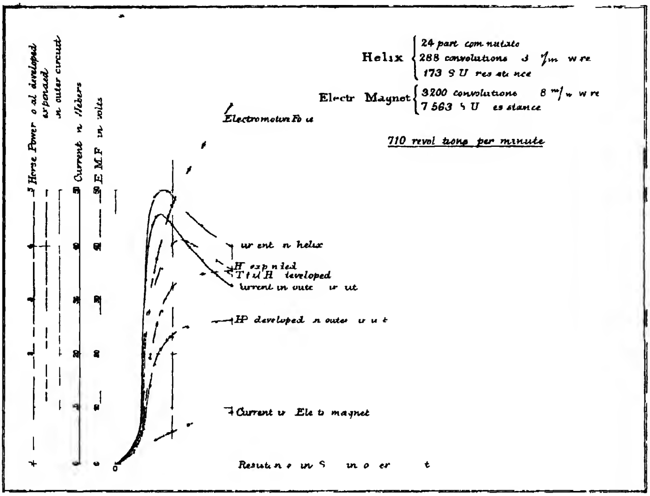
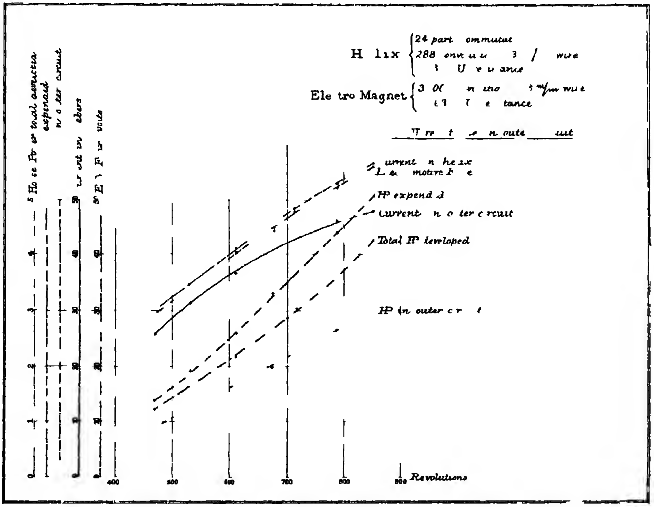
Nº 20



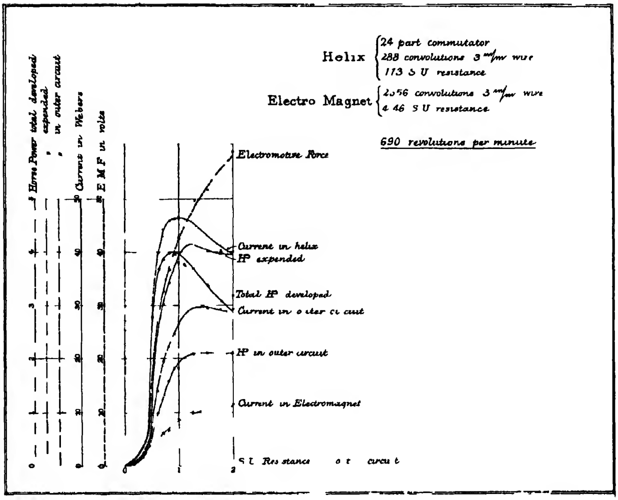
Helix
24 part commutator
288 convolutions
Electr Magnet
3200 convolutions
710 revolutions per minute
Electromotive Force
Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
HP developed in outer circuit
Current in Electro magnet

Nº 21

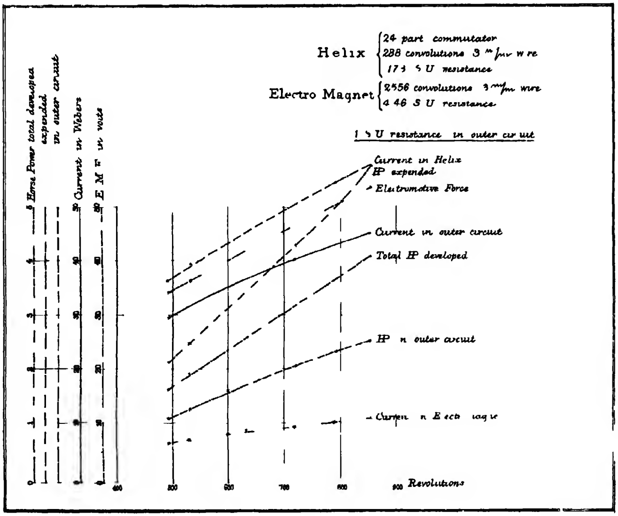


Helix
24 part commutator
288
Electro Magnet
HP expended
Total HP developed
HP in outer circuit
Revolutions
400
500
600
700
800
900

N° 22


Helix
24 part commutator
288 convolutions 3 m/m wire
113 S U resistance
Electro Magnet
2556 convolutions 3 m/m wire
4 46 S U resistance
690 revolutions per minute
Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
Electromotive Force
Current in helix
HP expended
Total HP developed
HP in outer circuit
Current in Electromagnet
S U Resistance

N° 23


Helix
24 part commutator
288 convolutions 3 m/m wire
S U resistance
Electro Magnet
2556 convolutions 3 m/m wire
4 46 S U resistance
1 S U resistance in outer circuit
Horse Power total developed
expended
in outer circuit
Current in Webers
E M F in volts
Current in Helix
HP expended
Electromotive Force
Current in outer circuit
Total HP developed
HP in outer circuit
500
600
700
800
900 Revolutions